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Generalized perfect graphs: Characterizations and inversion

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Abstract

Given a hereditary family of graphs $\mathscr{P}$ one defines the *$\mathscr{P}$-chromatic number* of a graph G (denoted $\chi_{\mathscr{P}}(G)$) to be the minimum size of a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that each V_i induces in G a member of $\mathscr{P}$. Define $\omega_{\mathscr{P}}(G)$ to equal $\max\{\chi_{\mathscr{P}}(K)\}$ where the maximum is taken over all cliques K in G. We say that G is $\chi_{\mathscr{P}}$-perfect provided $\chi_{\mathscr{P}}(H) = \omega_{\mathscr{P}}(H)$ for all induced subgraphs H of G and we denote the set of $\chi_{\mathscr{P}}$-perfect graphs by $\mathscr{P}^*$.

In this paper we discuss the following results:

(1) We give analogs of the Strong Perfect Graph Conjecture, that is, we find forbidden subgraph characterizations of $\mathscr{P}^*$ for various families $\mathscr{P}$.

(2) We show the central role played by the classes $\mathrm{Free}(K_n) = \{G: \omega(G) < n\}$ in finding $\mathscr{P}^*$ for *all* hereditary $\mathscr{P}$, and give a partial characterization of $(\mathrm{Free}(K_n))^*$ for $n \geq 3$.

(3) We consider the problem of inverting perfection: given a hereditary family $\mathscr{Q}$, find all hereditary $\mathscr{P}$ such that $\mathscr{P}^* = \mathscr{Q}$. We find conditions on $\mathscr{P}$ that are necessary and sufficient for $\mathscr{P}^* = \mathscr{Q}$. We then apply this "inverting perfection theorem" to a number of families $\mathscr{Q}$

1. Introduction

In this paper we define a generalization of graph perfection which we call "$\chi_{\mathscr{P}}$-perfection". We characterize the $\chi_{\mathscr{P}}$-perfect graphs for a number of hereditary properties $\mathscr{P}$, thereby proving analogs of the Strong Perfect Graph Conjecture. We also prove an inversion theorem which finds all hereditary families $\mathscr{P}$ whose $\chi_{\mathscr{P}}$-perfect graphs are a given set of graphs.

1.1. Background

Perfect graphs were first introduced by Berge in 1960, and since that time have received intense interest in the graph theory community. A graph G is *perfect* provided each of its induced subgraphs H satisfies the equation $\chi(H) = \omega(H)$ where χ is the chromatic number and ω is the clique number, i.e., the maximum number of pairwise adjacent vertices in H.

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One reason for this interest is that many families of graphs which naturally arise in applications (e.g., bipartite, chordal, comparability, interval) are indeed perfect. The considerable knowledge collected about these special classes has been used to give insight and evidence for new conjectures about perfect graphs. In particular, polynomial-time algorithms are known for finding the chromatic numbers of chordal, comparability and interval graphs [9], yet this problem is NP-complete for general graphs [8]. Grötschel et al. [10] have developed a polynomial-time algorithm for solving the chromatic number problem (i.e., given a graph G and an integer k, decide if $\chi(G) \leqslant k$)[1] for perfect graphs in general.

The foremost open problem concerning perfect graphs is Berge's Strong Perfect Graph Conjecture (SPGC), which characterizes the perfect graphs in terms of forbidden induced subgraphs [1].

Conjecture 1 (*SPGC*). The minimal imperfect graphs are the odd cycles on 5 or more vertices and the complements of odd cycles on 5 or more vertices.

In trying to prove the SPGC, authors have taken a variety of approaches. Lovász was able to prove that perfection is invariant under graph complementation, a result implied by the SPGC which is now known as the perfect graph theorem (PGT) [14].

Theorem 2 (PGT). *A graph G is perfect iff its complement $\bar{G}$ is perfect.*

Other authors have proved the SPGC for restricted families of graphs, such as (K_4–edge)-free, planar and claw-free graphs [17, 18, 21]. A different approach is to focus on the minimal imperfect graphs (which, according to the SPGC, would be the odd cycles, and their complements, on 5 or more vertices). A number of structural conditions for such graphs have been determined [2, 15, 16].

In this paper we define and study a generalization of graph perfection that we call "$\chi_{\mathscr{P}}$-perfection". Our definition brings together the area of perfect graphs with that of the $\mathscr{P}$-chromatic number (which simultaneously generalizes the chromatic number and such variants as vertex arboricity, vertex thickness and the cochromatic number). Cai and Corneil [5, 6] are studying a generalization of perfection which they call "i-perfection". Their i-perfection is the same as our $\chi_{\mathscr{P}}$-perfection when we restrict attention to the properties $\mathscr{P} = \mathrm{Free}(K_{i+1})$.

1.2. Overview of results

In Section 2 we define what it means for a graph to be (generalized) $\chi_{\mathscr{P}}$-perfect and collect some basic results about $\chi_{\mathscr{P}}$-perfection.

[1] And also the related clique problem, independent set problem and clique covering problem.

With that background we concentrate on answering the following questions.

(1) What is the set of perfect graphs? While this question is unresolved (see Conjecture 1), we *can* find the set of $\chi_{\mathscr{P}}$-perfect graphs (denoted $\mathscr{P}^*$) for a number of specific properties $\mathscr{P}$. For example, when $\mathscr{P} = \{\text{acyclic graphs}\}$ we have $\mathscr{P}^* = \{\text{chordal graphs}\}$ (see Theorem 10), i.e., a graph is "acyclic-perfect" if and only if it is chordal. Such characterizations are analogs of the SPGC. In Section 3 we list the theorems themselves; the proofs appear in [20].

(2) Can we characterize one class of generalized perfect graphs from another? Indeed we can; the $\mathscr{X}$-free restriction theorem (in Section 3) enables us to calculate $\mathscr{Q}^*$ given $\mathscr{P}^*$ for many hereditary families $\mathscr{Q} \subseteq \mathscr{P}$. We think of the properties $\mathscr{Q}$ as "restricted versions" of $\mathscr{P}$. The $\mathscr{X}$-free restriction theorem also singles out the properties $\text{Free}(K_n)$ for $n \geqslant 2$ as playing a key role in determining $\mathscr{P}^*$ for *all* hereditary properties $\mathscr{P}$. We discuss this connection and give a partial characterization of $(\text{Free}(K_n))^*$ for $n \geqslant 3$ (the case $n = 2$ is the SPGC).

(3) Can a given set of graphs $\mathscr{Q}$ be recognized as the set of $\chi_{\mathscr{P}}$-perfect graphs for some property $\mathscr{P}$? If so, what are *all* such properties $\mathscr{P}$? We call this the inverting perfection problem since it seeks to invert the function $*: \mathscr{P} \mapsto \mathscr{P}^*$. Returning to our example, if $\mathscr{Q} = \{\text{chordal graphs}\}$ then $\mathscr{P}^* = \mathscr{Q}$ if and only if $\mathscr{P} = \{\text{chordal graphs}\}$ or $\mathscr{P} = \{\text{chordal graphs}\} \cap \text{Free}(K_n)$ for some $n \geqslant 3$. Note that for $n = 3$ we recover the property $\mathscr{P} = \{\text{acyclic graphs}\}$. In Section 4 we solve the inverting perfection problem in general; the solution takes the same form as the one in our example although the restrictions on n can be different.

In Section 5 we discuss open problems and suggest directions for future research.

2. Generalized perfect graphs – preliminaries

2.1. Notation

We assume that the reader is familiar with the basic concepts in graph theory. Any definitions that are omitted here can be found in [3] or any other standard text on graph theory.

All graphs in this paper are assumed to be finite and undirected, with no loops or multiple edges and with nonempty vertex sets. We do not distinguish between isomorphic graphs.

A graph H is an *induced subgraph* of G, denoted $H \leqslant G$, if $V(H) \subseteq V(G)$ and $E(H)$ consists of exactly those edges of G with both endpoints in $V(H)$. If H is induced in G and $H \neq G$, then we write $H < G$; we write $G[X]$ to denote the induced subgraph of G with vertex set $X \subseteq V(G)$.

The *chromatic number* $\chi(G)$ is the minimum number of colors needed to color the vertex set $V(G)$ properly, i.e., so that no two adjacent vertices get the same color. Alternatively, $\chi(G)$ is the minimum size of a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ so that for each i, the induced subgraph $G[V_i]$ is edgeless. The *clique number* $\omega(G)$ is the number of vertices in a largest complete subgraph induced in G. A graph G is *perfect* provided $\chi(H) = \omega(H)$ for all induced subgraphs $H \leqslant G$.

We will refer frequently to graph *properties* $\mathscr{P}$ (also called *classes* or *families*), which are defined to be sets of graphs. A *nontrivial* property is one which is not equal to the empty set. We say a property $\mathscr{P}$ is *hereditary* if it is closed under taking induced subgraphs, i.e., if $G \in \mathscr{P}$ and $H \leqslant G$ together imply that $H \in \mathscr{P}$. All properties we consider will be hereditary and nontrivial.

If $\mathscr{P}$ is a hereditary property, G is called a *minimal forbidden* (or just *forbidden*) graph for $\mathscr{P}$ if $G \notin \mathscr{P}$ but $H \in \mathscr{P}$ for all proper induced subgraphs $H < G$. Denote the set of minimal forbidden graphs for $\mathscr{P}$ by Forb$(\mathscr{P})$; thus Forb$(\mathscr{P}) = \{G\colon G \notin \mathscr{P}$ but $H \in \mathscr{P}$ for all $H < G\}$. As an example, the family $\mathscr{P} = \{$acyclic graphs$\}$, has Forb$(\mathscr{P}) = \{C_r\colon r \geqslant 3\}$, where C_r is the cycle on r vertices. For a set of graphs $\mathscr{F}$ we also write Free$(\mathscr{F}) = \{G\colon F \not\leqslant G$ for all $F \in \mathscr{F}\}$. In particular, $\mathscr{F} =$ Forb$(\mathscr{P})$ implies that $\mathscr{P} =$ Free$(\mathscr{F})$, and $\mathscr{P} =$ Free$(\mathscr{F})$ implies that Forb$(\mathscr{P}) \subseteq \mathscr{F}$. A property $\mathscr{P}$ can be expressed as $\mathscr{P} =$ Free $(\mathscr{F})$ if and only if $\mathscr{P}$ is hereditary.

2.2. Basic definitions

A generalization of the chromatic number has been investigated by many authors [4, 11, 13].

Definition 3. The *$\mathscr{P}$-chromatic number* $\chi_{\mathscr{P}}(G)$ is the minimum size of a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ so that for each i we have $G[V_i] \in \mathscr{P}$.

The partition itself is called a *$\mathscr{P}$-coloring using k colors*. The vertices in V_i are considered to receive color i, and thus the condition $G[V_i] \in \mathscr{P}$ for all i translates to the restriction that "each color class induces a graph in $\mathscr{P}$". When $\mathscr{P}$ is the set of edgeless graphs, $\chi_{\mathscr{P}}$ reduces to the ordinary chromatic number. Other specific instances of $\chi_{\mathscr{P}}$ which have been studied include: vertex arboricity ($\mathscr{P} = \{$acyclic graphs$\}$), vertex thickness ($\mathscr{P} = \{$planar graphs$\}$), and cochromatic number ($\mathscr{P} =$ cliques $\cup$ edgeless graphs). For example, if $\mathscr{P} = \{$acyclic graphs$\}$ then the cycle C_4 cannot be $\mathscr{P}$-colored with just 1 color because C_4 is not acyclic. However, any partition of its vertex set into 2 nonempty parts yields a valid $\mathscr{P}$-coloring, thus $\chi_{\mathscr{P}}(C_4) = 2$.

We next generalize the clique number ω. Our definition is motivated by the observation that $\omega(G)$ is not only the number of vertices in a largest clique in G, but also the largest chromatic number of a clique in G, that is,

$$\omega(G) = \max\{\chi(K)\colon K \leqslant G \text{ and } K \text{ is a clique}\}.$$

The advantage to this latter definition is that it easily generalizes.

Definition 4. *The $\mathscr{P}$-clique number* of G, denoted $\omega_{\mathscr{P}}(G)$, is defined as

$$\omega(G) = \max\{\chi_{\mathscr{P}}(K)\colon K \leqslant G \text{ and } K \text{ is a clique}\}.$$

It is now natural to define the following generalization of perfection.

Definition 5. A graph G is *$\chi_{\mathscr{P}}$-perfect* if $\chi_{\mathscr{P}}(H) = \omega_{\mathscr{P}}(H)$ for all induced subgraphs $H \leqslant G$. We denote the set of $\chi_{\mathscr{P}}$-perfect graphs by $\mathscr{P}^*$.

If a graph G is *not* in $\mathscr{P}^*$, we say it is *$\chi_{\mathscr{P}}$-imperfect*. Moreover, if $G \notin \mathscr{P}^*$ but $H \in \mathscr{P}^*$ for all $H < G$, then G is a *minimal $\chi_{\mathscr{P}}$-imperfect* graph.

It is not hard to see that the $\mathscr{P}$-chromatic number and the $\mathscr{P}$-clique number are well-defined for all hereditary properties $\mathscr{P}$. Furthermore, if G is any graph, then $\chi_{\mathscr{P}}(G)$ and $\omega_{\mathscr{P}}(G)$ are integers between 1 and $|V(G)|$ (inclusive).

2.3. *Preliminary results*

In this section we collect some basic results about $\chi_{\mathscr{P}}$, $\omega_{\mathscr{P}}$ and $\mathscr{P}^*$ that will be used later in the paper. In most cases the results are easy consequences of definitions, and thus we omit the proofs. For a more complete treatment, see [20].

Proposition 6.
(a) *For the property $\mathscr{P} = \{$edgeless graphs$\}$, the definitions of $\chi_{\mathscr{P}}$, $\omega_{\mathscr{P}}$, and $\chi_{\mathscr{P}}$-perfection reduce to those of χ, ω, and perfection, respectively. Thus $\chi_{\mathscr{P}}$-perfection is indeed a generalization of (ordinary) graph perfection.*
(b) *For any hereditary property $\mathscr{P}$, if $H \leqslant G$ then $\chi_{\mathscr{P}}(H) \leqslant \chi_{\mathscr{P}}(G)$, and $\omega_{\mathscr{P}}(H) \leqslant \omega_{\mathscr{P}}(G)$.*
(c) *For all hereditary properties $\mathscr{P}$ and all graphs G, we have $\omega_{\mathscr{P}}(G) \leqslant \chi_{\mathscr{P}}(G)$.*
(d) *For all hereditary properties $\mathscr{P}$, the set $\mathscr{P}^*$* of *$\chi_{\mathscr{P}}$-perfect graphs is also hereditary.*
(e) *If $\mathscr{P}$ is a hereditary property and G is a graph, then $G \in \mathscr{P}$ if and only if $\chi_{\mathscr{P}}(G) = 1$.*
(f) *If $\mathscr{P}$ is a hereditary property then $\mathscr{P} \subseteq \mathscr{P}^*$, that is, all graphs in $\mathscr{P}$* are *$\chi_{\mathscr{P}}$-perfect.*
(g) *For all hereditary properties $\mathscr{P}$, we have $K_n \in \mathscr{P}^*$ for all integers $n \geqslant 1$.*
(h) *If $\mathscr{P}$ is a hereditary property, then $\mathscr{P}^* = \mathscr{P}$ if and only if the class $\mathscr{P}$ contains all cliques.*
(i) *If $\mathscr{P}$ is any hereditary property then $(\mathscr{P}^*)^* = \mathscr{P}^*$.*

2.3.1. *Computing $\omega_{\mathscr{P}}(G)$*

We conclude this section on preliminaries with a proposition that allows us to find $\omega_{\mathscr{P}}(G)$ directly from $\omega(G)$.

Definition 7. Let $\omega(\mathscr{P})$ be the number of vertices in a largest clique in hereditary $\mathscr{P}$, if such a clique exists, and write $\omega(\mathscr{P}) = \infty$ otherwise.

Thus if $\mathscr{P}$ contains all cliques then $\omega(\mathscr{P}) = \infty$ and if not, $\omega(\mathscr{P}) = m$ where $K_m \in \mathscr{P}$ but $K_{m+1} \notin \mathscr{P}$. Note that $\omega(\mathscr{P})$ is well-defined for all hereditary properties $\mathscr{P}$ because $K_{m+1} \notin \mathscr{P}$ implies that $K_M \notin \mathscr{P}$ for all $M \geqslant m+1$.

Proposition 8. *For any graph G,*
1. *If $\omega(\mathscr{P}) = \infty$ then $\omega_{\mathscr{P}}(G) = 1$.*
2. *If $\omega(\mathscr{P}) < \infty$, then $\omega_{\mathscr{P}}(G) = \lceil \omega(G)/\omega(\mathscr{P}) \rceil$.*

Proof. If $\omega(\mathscr{P}) = \infty$ then $K_m \in \mathscr{P}$ for all $m \geqslant 1$. Thus $\chi_{\mathscr{P}}(K_m) = 1$ for all $m \geqslant 1$ and $\omega_{\mathscr{P}}(G) = 1$ for all graphs G.

Otherwise, let $m = \omega(\mathscr{P})$. Recall that $\omega_{\mathscr{P}}(G) = \max\{\chi_{\mathscr{P}}(K)\}$ where the maximum is taken over all cliques K with $K \leqslant G$. This maximum is attained at K_n where K_n is a largest clique in G, because for all other cliques K in G, we have $K \leqslant K_n$ and thus $\chi_{\mathscr{P}}(K) \leqslant \chi_{\mathscr{P}}(K_n)$ (Proposition 6(b)). Therefore, $\omega_{\mathscr{P}}(G) = \chi_{\mathscr{P}}(K_n)$ with $n = \omega(G)$. To find $\chi_{\mathscr{P}}(K_n)$, note that in any $\mathscr{P}$-coloring of K_n, each color class might contain up to m vertices if $n \geqslant m$ ($K_m \in \mathscr{P}$), but no color class can contain more ($K_{m+1} \notin \mathscr{P}$). Therefore,

$$\omega_{\mathscr{P}}(G) = \chi_{\mathscr{P}}(K_n) = \lceil n/m \rceil = \lceil \omega(G)/\omega(\mathscr{P}) \rceil. \qquad \square$$

3. Strong perfect graph theorems

3.1. Finding $\mathscr{P}^*$

In this section we characterize $\mathscr{P}^*$ for a number of hereditary properties $\mathscr{P}$. Often our characterizations take the form of a list of the minimal $\chi_{\mathscr{P}}$-imperfect graphs. These theorems are analogs to the Strong Perfect Graph Conjecture (Conjecture 1) which asserts that the minimal imperfect graphs are the odd holes and odd anti-holes on 5 or more vertices.

The properties $\mathscr{P}$ for which we have succeeded in finding $\mathscr{P}^*$ have an important feature in common: each of the minimal $\chi_{\mathscr{P}}$-imperfect graphs has $\omega_{\mathscr{P}} = 1$. In contrast, when $\mathscr{P}$ is the set of edgeless graphs, $\overline{C_{2k+1}}$ is a minimal imperfect graph and has $\omega_{\mathscr{P}}(\overline{C_{2k+1}}) = \omega(\overline{C_{2k+1}}) = k$ for each $k \geqslant 2$. Thus we are motivated to make the following definition.

Definition 9. A hereditary property $\mathscr{P}$ is called *unit-based* if each minimal $\chi_{\mathscr{P}}$-imperfect graph G has $\omega_{\mathscr{P}}(G) = 1$.

Once a property $\mathscr{P}$ is classified as unit-based, we are given a lot of information about the minimal $\chi_{\mathscr{P}}$-imperfect graphs. In many unit-based cases, we have polynomial-time algorithms to list the minimal $\chi_{\mathscr{P}}$-imperfect graphs and to determine if a given graph is $\chi_{\mathscr{P}}$-perfect. Thus proving that a property $\mathscr{P}$ is unit-based is a big step towards actually finding $\mathscr{P}^*$. In [19, 20] we prove a generalization of the perfect graph theorem which holds for unit-based properties.

The proofs of the following theorems are omitted since many of them are lengthly and all are given in [20]. In [20], polynomial-time algorithms to determine membership in $\mathscr{P}^*$ are given for the properties of Theorems 12 and 13.

Theorem 10. *For the property $\mathscr{P} = \{$acyclic graphs$\}$, we have $\mathscr{P}^* = \{$chordal graphs$\}$.*

Theorem 11. *The property $\mathscr{P} = \{$unicyclic graphs$\}$ is unit-based, that is, all minimal $\chi_{\mathscr{P}}$-imperfect graphs have $\omega_{\mathscr{P}} = 1$. Moreover, the only minimal $\chi_{\mathscr{P}}$-imperfect graphs are those shown in Fig. 1.*

Note that in Fig. 1 the small dots indicate that the cycles may be of any length.

Theorem 12. *Let $\mathscr{P} = \{G: \Delta(G) \leqslant t\}$ for a fixed positive integer t. Then $\mathscr{P}$ is unit-based. Moreover, G is $\chi_{\mathscr{P}}$-perfect if and only if for every $v \in V(G)$ either $d(v) \leqslant t$ or v is simplicial.*

Note that if $t = 0$ we have $\mathscr{P} = \{G: \Delta(G) \leqslant 0\} = \{$edgeless graphs$\}$ and so $\mathscr{P}^*$ is the set of (ordinary) perfect graphs. Unfortunately, the proof of Theorem 12 breaks down in the case $t = 0$ and thus does not resolve the SPGC. Indeed its conclusion, $\mathscr{P}^* = \{G$: for all $v \in V$ either v is simplicial or v is an isolated vertex$\}$, is false in that case.

Theorem 13. *The following property is unit-based: $\mathscr{P} = \{G: f(|V(G)|, |E(G)|) \leqslant t\}$ where t is a fixed real number, and f is a function that satisfies*
- *$a < c$ and $b < d \Rightarrow f(a,b) < f(c,d)$,*
- *$a \leqslant c$ and $b \leqslant d \Rightarrow f(a,b) \leqslant f(c,d)$,*
- *$f(2,1) \leqslant t$ (i.e., $K_2 \in \mathscr{P}$).*

The property "bounded number of edges", that is, $\mathscr{P} = \{G: |E(G)| \leqslant t\}$ for $t \geqslant 1$, is an example of a family that satisfies the conditions of Theorem 13 (with $f(x,y) = y$). A proof of Theorem 13 in that instance is given in [19].

Theorem 14. *The property $\mathscr{P} = \{G$: the maximum eigenvalue of G is at most $t\} = \{G: \rho(A(G)) \leqslant t\}$ is unit-based for any fixed positive integer t.*

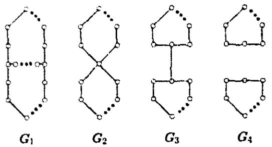

Fig. 1. Minimal $\chi_{\mathscr{P}}$-imperfect graphs for $\mathscr{P} = \{$unicyclic graphs$\}$.

It is not clear, a priori, that the property "maximum eigenvalue" of Theorem 14 is even hereditary. The fact that $\mathscr{P} = \{G: \rho(A(G)) \leqslant t\}$ is hereditary follows from the "interlacing eigenvalue theorem" in matrix theory which can be found as Theorem 4.3.8 in [12].

3.2. $\mathscr{X}$-free restrictions

In this section, we extend our characterizations of $\chi_{\mathscr{P}}$-perfect graphs. In particular, we characterize the $\chi_{\mathscr{Q}}$-perfect graphs, where $\mathscr{Q}$ is a variant of a property $\mathscr{P}$ whose perfect graphs are already known. Recall that for a set of graphs $\mathscr{X}$ we write $\mathrm{Free}(\mathscr{X}) = \{G: H \nleqslant G \text{ for all } H \in \mathscr{X}\}$.

Theorem 15 ($\mathscr{X}$-free restriction). *Let $\mathscr{P}$ be a nontrivial, hereditary property, and let $\mathscr{X}$ be a set of graphs that satisfies:*
(1) $\mathscr{X} \subseteq \mathscr{P}$, and
(2) $K_m \notin \mathscr{X}$ for all m.
If $\mathscr{Q} = \mathscr{P} \cap \mathrm{Free}(\mathscr{X})$ then $\mathscr{Q}^* = \mathscr{P}^* \cap \mathrm{Free}(\mathscr{X})$ that is,

$$G \text{ is } \chi_{\mathscr{Q}}\text{-perfect} \Leftrightarrow G \text{ is } \chi_{\mathscr{P}}\text{-perfect and } G \in \mathrm{Free}(\mathscr{X}).$$

Condition (1) is an "irredundancy" condition; there is no point in forbidding something that is already excluded. Thus it places no substantive restriction on $\mathscr{X}$. Condition (2) ensures that $\omega(\mathscr{P}) = \omega(\mathscr{Q})$ and therefore that $\omega_{\mathscr{P}}(G) = \omega_{\mathscr{Q}}(G)$ for any graph G (see Proposition 8).

The proof of Theorem 15 is not too difficult and can be found in [19].

Corollary 16. *If $\mathscr{P}$ and $\mathscr{Q}$ are nontrivial hereditary properties with $\omega(\mathscr{P}) \leqslant \omega(\mathscr{Q})$ then $(\mathscr{P} \cap \mathscr{Q})^* = \mathscr{P}^* \cap \mathrm{Free}(\mathscr{X})$ where $\mathscr{X} = \{G: G \in \mathscr{P} \text{ and } G \notin \mathscr{Q}\}$.*

Note that the condition "$\omega(\mathscr{P}) \leqslant \omega(\mathscr{Q})$" is not restrictive since for any two properties $\mathscr{P}$ and $\mathscr{Q}$ we have either $\omega(\mathscr{P}) \leqslant \omega(\mathscr{Q})$ or $\omega(\mathscr{Q}) \leqslant \omega(\mathscr{P})$.

Proof. Write $\mathscr{P} \cap \mathscr{Q} = \mathscr{P} \cap \mathrm{Free}(\mathscr{X})$ where $\mathscr{X} = \{G: G \in \mathscr{P} \text{ and } G \notin \mathscr{Q}\}$. There cannot be any cliques in $\mathscr{X}$, since any clique in $\mathscr{P}$ is also in $\mathscr{Q}$ by the assumption $\omega(\mathscr{P}) \leqslant \omega(\mathscr{Q})$. In addition, $\mathscr{X} \subseteq \mathscr{P}$. Therefore, Theorem 15 applies and yields the desired result: $(\mathscr{P} \cap \mathscr{Q})^* = \mathscr{P}^* \cap \mathrm{Free}(\mathscr{X})$. $\square$

3.3. The classes K_n-free

Our ambitious goal is to find $\mathscr{P}^*$ for all hereditary classes $\mathscr{P}$. In doing so we should take advantage of the $\mathscr{X}$-free restriction theorem (Theorem 15) which finds the $\chi_{\mathscr{Q}}$-perfect graphs when $\mathscr{Q}$ is a variant of $\mathscr{P}$ and $\mathscr{P}^*$ is known. The following two-step method would characterize the $\chi_{\mathscr{P}}$-perfect graphs for all hereditary properties $\mathscr{P}$. First we find a minimal set $\mathscr{B}$ of base properties, so that if $\mathscr{P}^*$ is known for each $\mathscr{P} \in \mathscr{B}$, then $\mathscr{P}^*$ can be computed (via Theorem 15) for all hereditary properties $\mathscr{P}$. Then we find $\mathscr{P}^*$ for every $\mathscr{P} \in \mathscr{B}$.

The first step is not too difficult. In the next theorem we show that the only such base set $\mathscr{B}$ is the set of properties $\{\text{Free}(K_n)\text{: } n \geqslant 2\}$ together with the property $\mathscr{A} = \{\text{all graphs}\}$. However, finding $(\text{Free}(K_n))^*$ appears to be quite difficult. In the case $n = 2$ we have $\text{Free}(K_n) = \{\text{edgeless graphs}\}$, and thus finding $\mathscr{P}_2^*$ would resolve the Strong Perfect Graph Conjecture. We devote the main portion of this section to giving some partial characterizations of $(\text{Free}(K_n))^*$

3.3.1. *The significance of finding* $(\text{Free}(K_n))^*$

Theorem 17. *Let $\mathscr{B}$ be the set of properties $\{\text{Free}(K_n)\text{: } n \geqslant 2\} \cup \{\mathscr{A}\}$, where $\mathscr{A} = \{$all graphs$\}$. Then all hereditary properties $\mathscr{Q}$ are either in $\mathscr{B}$ or can be written $\mathscr{Q} = \mathscr{P} \cap \text{Free}(\mathscr{X})$ where $\mathscr{P} \in \mathscr{B}$ and $\mathscr{X}$ satisfies conditions* (1) *and* (2) *of Theorem* 15. *Moreover, none of the properties in $\mathscr{B}$ can be expressed as such an $\mathscr{X}$-free restriction of any other property.*

Proof. Let $\mathscr{Q}$ be a hereditary property. If $\omega(\mathscr{Q}) = \infty$, then $\mathscr{Q}$ contains all cliques, and $\mathscr{Q} = \mathscr{A} \cap \text{Free}(\mathscr{X})$ where $\mathscr{A} = \{\text{all graphs}\}$ and $\mathscr{X} = \{G\text{: } G \notin \mathscr{Q}\}$. Clearly $\mathscr{X} \subseteq \mathscr{A}$, and $K_m \notin \mathscr{X}$ for all m, because $K_m \in \mathscr{Q}$ for all m. Hence in this case, $\mathscr{X}$ satisfies conditions (1) and (2) of Theorem 15.

Otherwise, $\omega(\mathscr{Q}) = n - 1$ for some integer $n \geqslant 2$, which means that $K_{n-1} \in \mathscr{Q}$ but $K_n \notin \mathscr{Q}$. Thus $\mathscr{Q} = \text{Free}(K_n) \cap \text{Free}(\mathscr{X})$ where $\mathscr{X} = \{G\text{: } G \in \text{Free}(K_n) \text{ and } G \notin \mathscr{Q}\}$. It is clear that $\mathscr{X} \subseteq \text{Free}(K_n)$. Furthermore, $K_m \notin \text{Free}(K_n)$ for $m \geqslant n$ and $K_m \in \mathscr{Q}$ for $m < n$, hence there are no cliques in $\mathscr{X}$. So again the conditions of Theorem 15 are met, and the main part of Theorem 17 is proved.

Recall that if $\mathscr{Q}$ is an $\mathscr{X}$-free restriction of $\mathscr{P}$, as in Theorem 15, then $\omega(\mathscr{Q}) = \omega(\mathscr{P})$, and $\mathscr{Q} \subseteq \mathscr{P}$. The property $\text{Free}(K_n)$ has $\omega(\text{Free}(K_n)) = n - 1$. Any other hereditary property $\mathscr{P}$ with $\omega(\mathscr{P}) = n - 1$ is a subset of $\text{Free}(K_n)$. Therefore, Free (K_n) cannot be written as an $\mathscr{X}$-free restriction of another property. Similarly, the property $\mathscr{A} = \{$all graphs$\}$ is not a subset of any other property, so it too cannot be written as an $\mathscr{X}$-free restriction. This justifies the last sentence of the theorem. □

Proposition 6(h) tells us that $\mathscr{A}^* = \mathscr{A}$, and that in fact $\mathscr{Q}^* = \mathscr{Q}$ for all $\mathscr{X}$-free restrictions of $\mathscr{A}$. Hence all that remains in our two-step approach is to find $(\text{Free}(K_n))^*$ for $n \geqslant 2$. We state this as a corollary.

Corollary 18. *In order to characterize $\mathscr{P}^*$ for every hereditary property $\mathscr{P}$, it suffices to find* $(\text{Free}(K_n))^*$ *for all $n \geqslant 2$.*

We have not been able to find $(\text{Free}(K_n))^*$ for any $n \geqslant 2$, and this appears to be a difficult problem. In particular, finding $(\text{Free}(K_2))^*$ would resolve the Strong Perfect

Graph Conjecture (SPGC). Nonetheless, we have some partial results in characterizing $(\mathrm{Free}(K_n))^*$ for $n \geqslant 2$.

In order to avoid double subscripts, we use the notation $\chi_n(G)$ to mean $\chi_{\mathscr{P}}(G)$, and $\omega_n(G)$ to mean $\omega_{\mathscr{P}}(G)$, where $\mathscr{P} = \mathrm{Free}(K_n)$.

3.3.2. Partial characterizations of $(\mathrm{Free}(K_n))^*$

We will see in Corollary 24 that if the SPGC is true, a χ_n-imperfect graph *must* contain an odd hole C_r or an odd anti-hole $\overline{C_r}$. Thus we begin our study of the class $(\mathrm{Free}(K_n))^*$ by focusing on the odd anti-holes. The next lemma tells us exactly which complements of cycles $(\overline{C_r})$ are in $(\mathrm{Free}(K_n))^*$ and the theorem following it gives a more general result. After that we state a theorem concerning the relationship between $(\mathrm{Free}(K_n))^*$ and $(\mathrm{Free}(K_m))^*$ and discuss its connection to the SPGC.

Lemma 19. *The anti-hole $\overline{C_r}$ is a minimal χ_n-imperfect graph if $r \equiv 1 \bmod (2n-2)$ and $r > 2n-1$, and is χ_n-perfect otherwise.*

Note that in the case $n = 2$, Lemma 19 states that $\overline{C_r}$ is a minimal imperfect graph if and only if r is odd and greater than 3 (which agrees with the SPGC).

To prove Lemma 19, it suffices to show:

1. $\chi_n(\overline{C_r}) \neq \omega_n(\overline{C_r})$ iff $r \equiv 1 \bmod (2n-2)$ and $r > 2n-1$.
2. For all r and all $H < \overline{C_r}$, we have $\chi_n(H) = \omega_n(H)$.

Proof of Lemma 19. Recall that $\omega(\mathrm{Free}(K_n)) = n-1$, so $\omega_n(G) = \lceil \omega(G)/(n-1) \rceil$ for all graphs G.

Part 1: If $r \leqslant 2n-1$ then the largest clique in $\overline{C_r}$ is at most K_{n-1}. In this case, $\overline{C_r} \in \mathrm{Free}(K_n)$ and so $\overline{C_r}$ is χ_n-perfect by Proposition 6(f).

Now assume $r > 2n-1$. In a χ_n-coloring of $\overline{C_r}$, each color class can have at most $2n-2$ vertices, with equality if the vertices are taken consecutively around the cycle, so

$$\chi_n(\overline{C_r}) = \left\lceil \frac{r}{2n-2} \right\rceil. \tag{1}$$

The maximum clique in $\overline{C_r}$ is $\lfloor r/2 \rfloor$ and thus

$$\omega_n(\overline{C_r}) = \left\lceil \left\lfloor \frac{r}{2} \right\rfloor \cdot \frac{1}{n-1} \right\rceil. \tag{2}$$

We now compare Eqs. (1) and (2). Write $r = (2n-2)q + s$ where $q \geqslant 1$ and $0 \leqslant s < 2n-2$. Then

$$\chi_n(\overline{C_r}) = \left\lceil \frac{r}{2n-2} \right\rceil = \left\lceil \frac{(2n-2)q+s}{2n-2} \right\rceil = q + \left\lceil \frac{s}{2n-2} \right\rceil = \begin{cases} q & \text{if } s = 0, \\ q+1 & \text{otherwise,} \end{cases}$$

and

$$\omega_n(\overline{C_r}) = \left\lceil \left\lfloor \frac{r}{2} \right\rfloor \cdot \frac{1}{n-1} \right\rceil = \left\lceil \left\lfloor \frac{(2n-2)q+s}{2} \right\rfloor \cdot \frac{1}{n-1} \right\rceil$$
$$= \left\lceil \left((n-1)q + \left\lfloor \frac{s}{2} \right\rfloor \right) \cdot \frac{1}{n-1} \right\rceil$$
$$= \left\lceil q + \left\lfloor \frac{s}{2} \right\rfloor \cdot \frac{1}{n-1} \right\rceil = \begin{cases} q & \text{if } s = 0 \text{ or } 1, \\ q+1 & \text{otherwise.} \end{cases}$$

The quantities $\chi_n(\overline{C_r})$ and $\omega_n(\overline{C_r})$ are unequal exactly when $s = 1$. Thus if $r > 2n-1$, we have $\chi_n(\overline{C_r}) \neq \omega_n(\overline{C_r})$ iff $r \equiv 1 \bmod (2n-2)$. This completes the proof of Part 1.

Part 2: Let $H < \overline{C_r}$, so $H \leqslant \overline{P_r}$ and thus $\overline{H} \leqslant P_r$. Let the components of $\overline{H}$ have vertex sets $V_1, V_2, \ldots, V_k$. [Thus $V_1 \cup V_2 \cup \cdots \cup V_k = V(\overline{H}) = V(H)$.]

Claim 1.

$$\omega_n(H) = \left\lceil \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot \frac{1}{n-1} \right\rceil.$$

Proof. For each i, the largest clique in $H[V_i]$ has $\lceil |V_i|/2 \rceil$ vertices. For $i \neq j$, every vertex of V_i is adjacent to every vertex of V_j in H. Therefore,

$$\omega(H) = \sum_{i=1}^{k} \lceil |V_i|/2 \rceil$$

and so

$$\omega_n(H) = \left\lceil \frac{\omega(H)}{n-1} \right\rceil = \left\lceil \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot \frac{1}{n-1} \right\rceil . \quad \square.$$

Claim 2.

$$\chi_n(H) \leqslant \left\lceil \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot \frac{1}{n-1} \right\rceil.$$

Proof. List the vertices of H in the following order:

$$x_{1,1}, x_{1,2}, \ldots, x_{1,|V_1|}, \ldots, x_{k,1}, x_{k,2} \ldots, x_{k,|V_k|},$$

where for each i, the vertices in V_i are listed so that $x_{i,1} \sim x_{i,2} \sim \cdots \sim x_{i,|V_i|}$ in $\overline{H} \leqslant P_r$. Place a "dummy" vertex after the last vertex of V_i whenever $|V_i|$ is odd.

Color the vertices of H as follows: put the first $2n-2$ vertices on the list (including dummies) in color class 1, the next $2n-2$ vertices in color class 2, etc., until all the vertices are used. This gives a valid K_n-free coloring, because in each color class, at most half the vertices are in a clique together, i.e., the largest clique has at most $(2n-2)/2 = n-1$ vertices.

To count the number of vertices in the list (including dummies), we add the term $|V_i|$, if $|V_i|$ is even, and the term $|V_i| + 1$, if $|V_i|$ is odd, for $1 \leqslant i \leqslant k$. Thus the total number of vertices is $\sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot 2$. Since each color class receives $2n - 2$ vertices (with the possible exception of the last one), the number of colors used is

$$\left\lceil \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot 2 \cdot \frac{1}{2n-2} \right\rceil .$$

Therefore,

$$\chi_n(H) \leqslant \left\lceil \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot \frac{1}{n-1} \right\rceil$$

as desired. □

Combining the results of Claims 1 and 2 gives the inequality $\chi_n(H) \leqslant \omega_n(H)$. The reverse inequality comes from Proposition 6(c), hence we get the equality desired in Part 2. This completes the proof of Lemma 19. □

Definition 20. For graphs G and H with disjoint vertex sets, the *join* $G \vee H$ of G and H is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{(x, y): x \in V(G) \text{ and } y \in V(H)\}$.

In Lemma 19 we determined exactly which odd anti-holes $\overline{C_r}$ are minimal χ_n-imperfect graphs. The following theorem is a more general result which shows that any sufficiently large odd anti-hole can be made to be a minimal χ_n-imperfect graph by joining it to an appropriate sized clique. The result was inspired by L. Cai's observation (personal communication) that the graph $\overline{C_7} \vee K_1$ is a minimal χ_3-imperfect graph.

Theorem 21. *Let $n \geqslant 2$ be any integer, r be an odd integer, and let $r' = r \bmod (2n-2)$, so that r' is odd with $1 \leqslant r' < 2n - 2$. If $r > 2n - 1$ then the graph $\overline{C_r} \vee K_s$ is a minimal χ_n-imperfect graph, where*

$$s = \begin{cases} 0 & \text{if } r' = 1, \\ n - (r'+1)/2 & \text{if } r' > 1. \end{cases}$$

If $r \leqslant 2n - 1$ (and r odd) then $\overline{C_r} \vee K_l$ is χ_n-imperfect for all $l > 0$.

Note that if a graph is a minimal χ_n-imperfect graph, then all its proper induced subgraphs are χ_n-perfect, and all graphs it is induced in are χ_n-imperfect. Hence, as the following example illustrates, Theorem 21 can be used to determine *exactly* which graphs of the form $\overline{C_r} \vee K_l$ are χ_n-perfect for $r > 2n - 1$.

Example 22. Consider the property Free(K_{17}) (i.e., $n = 17$) and the odd anti-hole $\overline{C_r}$ where $r = 41$. Then $2n - 2 = 32$ and $r \equiv 9 \pmod{2n-2}$, so $r' = 9$. Thus $s = 17 - 5 = 12$, and by Theorem 21 we know that $\overline{C_{41}} \vee K_{12}$ is a minimal χ_{17}-imperfect graph. We conclude that $\overline{C_{41}} \vee K_l$ is χ_{17}-perfect if and only if $l < 12$.

Proof of Theorem 21. Recall that $\omega(\mathrm{Free}(K_n)) = n - 1$, so $\omega_n(G) = \lceil \omega(G)/(n-1) \rceil$ for all graphs G. If r is odd and $r \leqslant 2n - 1$ then it is not too hard to modify the argument in the proof of Lemma 19 (Part 2) to show that $\overline{C_r} \vee K_l$ is χ_n-perfect for all $l \geqslant 0$. Hence we restrict attention to the case of $r > 2n - 1$.

If $r' = 1$, then $s = 0$ and so $\overline{C_r} \vee K_s = \overline{C_r}$. In this case, the hypothesis of Lemma 19 are satisfied, and so $\overline{C_r} \vee K_s$ is a minimal χ_n-imperfect graph by the conclusion of that lemma.

Now assume $3 \leqslant r' < 2n - 2$ and let $s = n - (r' + 1)/2$. Write $r = (2n - 2)q + r'$ where $q \geqslant 1$ is an integer. In the interest of shortening notation, let $G = \overline{C_r} \vee K_s$.

To prove Theorem 21, it suffices to show:

1. $\chi_n(G) \neq \omega_n(G)$, and
2. $\chi_n(H) = \omega_n(H)$ for every $H < G$.

Proof of Part 1. A largest clique in $G = \overline{C_r} \vee K_s$ consists of a largest clique in $\overline{C_r}$ together with the entire clique K_s. Thus

$$\omega(G) = \lfloor r/2 \rfloor + s = \left\lfloor \frac{(2n-2)q + r'}{2} \right\rfloor + s = \frac{(2n-2)q + r' - 1}{2} + s$$

since r' is odd. Then

$$\omega_n(G) = \left\lceil \frac{\omega(G)}{n-1} \right\rceil = \left\lceil \frac{1}{n-1}\left(\frac{(2n-2)q + r' - 1}{2} + n - \frac{r'+1}{2}\right) \right\rceil$$

$$= \left\lceil \frac{1}{n-1}((n-1)q + (n-1)) \right\rceil = \lceil q + 1 \rceil = q + 1.$$

We next consider $\chi_n(G)$. In any χ_n-coloring of G, each color class can have at most $2n - 2$ vertices, with equality if and only if the $2n - 2$ vertices are taken consecutively around the anti-hole $\overline{C_r}$. Since $r = (2n - 2)q + r'$ with $3 \leqslant r' < 2n - 2$, we can form $q \geqslant 1$ such classes. We have r' vertices remaining in the $\overline{C_r}$, which induce a $\overline{P_{r'}}$ in G, and all s vertices from the clique K_s which still need to be colored. The largest clique in the graph induced by these remaining vertices has $\lceil r'/2 \rceil + s$ vertices. Since r' is odd, $\lceil r'/2 \rceil + s = (r' + 1)/2 + s = n$. Thus the remaining vertices induce a graph which is *not* in Free(K_n), and hence must be put into at least two color classes. Therefore, $q + 2 \leqslant \chi_n(G) \neq \omega_n(G) = q + 1$. This completes the proof of Part 1.

Proof of Part 2. Let H be a graph strictly induced in $G = \overline{C_r} \vee K_s$. We will show that $\chi_n(H) = \omega_n(H)$. Recall that $r = (2n - 2)q + r'$ with $3 \leqslant r' < 2n - 2$ and $s = n - (r' + 1)/2$. Thus $1 \leqslant s \leqslant n - 2$.

Case 1: $H = \overline{C_r} \vee K_t$ *with* $0 \leqslant t < s \leqslant n-2$. In this case, H is formed by deleting one or more vertices from the clique K_s in G, and not deleting any of the vertices from the anti-hole $\overline{C_r}$. Note that the inequalities $0 \leqslant t < s \leqslant n-2$ imply that $-(n-2) \leqslant t-s < 0$ and thus

$$-1 < \frac{t-s}{n-1} < 0. \tag{3}$$

A largest clique in H consists of a largest clique in $\overline{C_r}$ together with the entire K_t. Thus

$$\begin{aligned}\omega(H) &= \left\lfloor \frac{r}{2} \right\rfloor + t = \left\lfloor \frac{(2n-2)q + r'}{2} \right\rfloor + t = (n-1)q + \frac{r'-1}{2} + t \\ &= (n-1)q + \frac{r'+1}{2} - 1 + t = (n-1)q + n - s - 1 + t \\ &= (n-1)(q+1) + t - s.\end{aligned}$$

Hence

$$\omega_n(H) = \left\lceil \frac{\omega(H)}{n-1} \right\rceil = \left\lceil q + 1 + \frac{t-s}{n-1} \right\rceil = q+1,$$

where the final equality is justified by Eq. (3).

Next we compute $\chi_n(H)$. We color the vertices of H as we did in the computation of $\chi_n(G)$ in Part 1 of this proof. First we color $(2n-2)q$ vertices of the anti-hole $\overline{C_r}$ using q colors. What remain are r' consecutive vertices in $\overline{C_r}$ and the t vertices of the clique K_t. The largest number of vertices which induce a clique in this remaining portion is

$$\lceil r'/2 \rceil + t = \frac{1+r'}{2} + t = n - s + t \leqslant n-1$$

since r' is odd, and $t < s$. Hence the remaining vertices induce a graph in Free(K_n) and can be colored with just 1 color. Thus $\chi_n(H) = q + 1 = \omega_n(H)$.

Case 2: $H = H' \vee K_t$ *where* $H' \leqslant \overline{P_r}$ *and* $t \leqslant s$. In this case, H is formed from G by deleting one or more of the vertices in the odd anti-hole $\overline{C_r}$ and perhaps some of the vertices in the clique K_s. Note that H' has the same form as the graph H in Part 2 of the proof of Lemma 19. As in that proof, we let the components of $\overline{H'} \leqslant P_r$ have vertex sets $V_1, V_2, \ldots, V_k$, and obtain $\omega(H') = \sum_{i=1}^{k} \lceil |V_i|/2 \rceil$. Write

$$2 \cdot \omega(H') = \sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot 2 = m(2n-2) + l$$

with $m \geqslant 0$ and l an even number satisfying $0 \leqslant l \leqslant 2n-3$. Then $\omega(H') = m(n-1) + l/2$. Since $\omega(H) = \omega(H') + \omega(K_t)$, we have

$$\omega(H) = m(n-1) + l/2 + t \quad \text{and} \quad \omega_n(H) = m + \left\lceil \frac{l/2 + t}{n-1} \right\rceil.$$

Now write $t = (n-1)u + t'$ with $0 \leqslant t' \leqslant n-2$. Note that

$$l/2 + t' \leqslant \frac{2n-3}{2} + n - 2 < 2n - 3. \tag{4}$$

Hence

$$\omega_n(H) = m + u + \left\lceil \frac{l/2 + t'}{n-1} \right\rceil = \begin{cases} m+u & \text{if } l = t' = 0, \\ m+u+1 & \text{if } 0 < l/2 + t' \leqslant n-1, \\ m+u+2 & \text{if } n-1 < l/2 + t' \leqslant 2n-4. \end{cases}$$

Next we compute $\chi_n(H)$. As in the proof of Claim 2 we list the vertices of H' and put in dummy vertices as necessary. There are $\sum_{i=1}^{k} \lceil |V_i|/2 \rceil \cdot 2 = m(2n-2) + l$ vertices on the list, including dummies. Color the first $m(2n-2)$ of them using m colors as discussed in the proof of Claim 2. Next color $(n-1)u$ of the vertices in the clique K_t using u colors (i.e., each color class induces a K_{n-1} which is in Free(K_n)). So far we have used $m+u$ colors, and we are left with $l \leqslant 2n-3$ consecutive vertices (including dummies) from our list of vertices in $\overline{H'} \leqslant P_r$, and $t' \leqslant n-2$ vertices from the clique K_t.

By Eq. (4) we know that $l/2 + t' \leqslant 2n-4$, hence we need only consider the following three cases.

- If $l = t' = 0$ then there are no vertices remaining, and $\chi_n(H) \leqslant m + u = \omega_n(H)$.
- If $0 \leqslant l/2 + t' \leqslant n-1$, then the remaining vertices induce a graph which is in Free(K_n) (i.e., its largest clique has size $\lceil l/2 \rceil + t' \leqslant l/2 + t' + 1/2 < n$). Hence $\chi_n(H) \leqslant m + u + 1 = \omega_n(H)$.
- If $n-1 < l/2 + t' \leqslant 2n-4$, then $\chi_n(H) \leqslant m+u+2$ since giving the l vertices of H' one color, and the t' vertices of the clique a second color produces a valid coloring. Again $\chi_n(H) \leqslant \omega_n(H)$.

In all cases we have $\chi_n(H) \leqslant \omega_n(H)$. The reverse inequality follows from Proposition 6(c), thus $\chi_n(H) = \omega_n(H)$ and we have completed the proof of Theorem 21. □

The next theorem, which is due to Cai and Corneil [6], tells us when χ_n-perfect graphs are χ_m-perfect.

Theorem 23 (Cai and Corneil [6]). *For all integers $n, m \geqslant 2$,*

$$(\text{Free}(K_n))^* \subseteq (\text{Free}(K_m))^* \quad \textit{iff} \quad (n-1)|(m-1).$$

Moreover, if $(n-1)|(m-1)$ but $m \neq n$, then the inclusion $(\text{Free}(K_n))^* \subset (\text{Free}(K_m))^*$ *is strict.*

The complete proof of Theorem 23 can be found in [20, 6]. Note that for $n = 2$ the property Free(K_n) generates the (ordinary) perfect graphs. Thus Theorem 23 has the following important corollary.

Corollary 24. *If G is a perfect graph, then G is χ_n-perfect for all $n \geqslant 2$.*

When Corollary 24 is combined with the $\mathscr{X}$-free restriction theorem (Theorem 15), we can determine whether a perfect graph is $\chi_{\mathscr{Q}}$-perfect for *any* hereditary property $\mathscr{Q}$. This is particularly useful if $\mathscr{Q}^*$ is not known.

Corollary 25 (When is a perfect graph $\chi_{\mathscr{P}}$-perfect?). *Let $\mathscr{Q}$ be any hereditary property and G be any perfect graph.*

1. *If $\omega(\mathscr{Q}) = \infty$ then $G \in \mathscr{Q}^* \Leftrightarrow G \in \mathscr{Q}$.*
2. *If $\omega(\mathscr{Q}) = n - 1 < \infty$, then write $\mathscr{Q} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{X})$ where $\mathscr{X} \subseteq \mathrm{Free}(K_n)$ and $K_m \notin \mathscr{X}$ for all m. In this case, $G \in \mathscr{Q}^* \Leftrightarrow G \in \mathrm{Free}(\mathscr{X})$.*

Proof. The first part follows from Proposition 6(h). For the second part, let G be a perfect graph and $\mathscr{Q}$ be as in the hypothesis of Corollary 25. Then $\mathscr{Q}^* = (\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{X})$ by Theorem 15. Since $G \subseteq \{\text{perfect graphs}\} = (\mathrm{Free}(K_2))^*$, Theorem 23 implies that $G \in (\mathrm{Free}(K_m))^*$ for all $m \geqslant 2$. Hence $G \in \mathscr{Q}^* \Leftrightarrow G \in \mathrm{Free}(\mathscr{X})$. $\square$

Another important consequence of Theorem 23 is that it may help us to find χ_n-imperfect graphs. If we assume the SPGC is true, then all χ_n-imperfect graphs *must* have an induced odd hole or odd anti-hole on 5 or more vertices. The minimal χ_n-imperfect graphs of Theorem 21 do indeed contain odd anti-holes. In fact, for each $n \geqslant 2$ and each odd $r > 2n - 1$, there is a minimal χ_n-imperfect graph containing $\overline{C}_r$. On the other hand, if we could find a minimal χ_n-imperfect graph with *no* induced odd hole and *no* induced odd anti-hole, then we would have a counterexample to the SPGC.

4. Inverting perfection

We can think of the $*$ operator as function $*: \mathscr{P} \mapsto \mathscr{P}^*$ whose domain is the set of nonempty hereditary properties $\mathscr{P}$. The image of $*$ is a set consisting of properties $\mathscr{Q}$ that are hereditary (by Proposition 6(d)) and have $\omega(\mathscr{Q}) = \infty$ (by Proposition 6(g)). In fact, *all* such properties are in the image of $*$ since $\mathscr{Q}^* = \mathscr{Q}$ for hereditary properties $\mathscr{Q}$ with $\omega(\mathscr{Q}) = \infty$ (Proposition 6(h)).

However, $*$ is *not* a 1–1 function: by Theorem 10 we know that $\{\text{acyclic graphs}\}^* = \{\text{chordal graphs}\}$, and by Proposition 6(d) we know that $\{\text{chordal graphs}\}^* = \{\text{chordal graphs}\}$. Thus we consider the problem of finding the inverse image under $*$ of a family $\mathscr{Q}$.

In Section 3 we examined questions of the form: given a hereditary property $\mathscr{P}$, find $\mathscr{P}^*$. Now we consider the inverse problem.

Inverting perfection problem. Given a hereditary class $\mathscr{Q}$ with $\omega(\mathscr{Q}) = \infty$, find all hereditary properties $\mathscr{P}$ with $\mathscr{P}^* = \mathscr{Q}$.

In the next section we present the inverting perfection theorem, which characterizes those properties $\mathscr{P}$ with $\mathscr{P}^* = \mathscr{Q}$. Afterwards we apply it to many specific classes $\mathscr{Q}$.

4.1. The inverting perfection theorem

Theorem 26 (Inverting perfection). *Let $\mathscr{Q}$ and $\mathscr{P}$ be hereditary classes of graphs so that $\mathscr{Q}$ contains all cliques, and let $\mathscr{F}$ be the set* Forb($\mathscr{Q}$) *of minimal forbidden graphs for $\mathscr{Q}$. Then $\mathscr{P}^* = \mathscr{Q}$ if and only if either*

- $\mathscr{P} = \mathscr{Q}$, *or*
- $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathscr{Q}$ *and $n \geqslant 2$ satisfies*
  1. $\mathscr{Q} \subseteq (\mathrm{Free}(K_n))^*$, *and*
  2. *for all $F \in \mathscr{F}$, either $F \in \mathrm{Free}(K_n)$ or $F \notin (\mathrm{Free}(K_n))^*$.*

Proof. Recall our notation $\mathscr{Q} = \mathrm{Free}(\mathscr{F})$ where $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ is the minimal set of graphs not in $\mathscr{Q}$. Hence

$$\mathrm{Free}(K_n) \cap \mathscr{Q} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{F}) = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)). \tag{5}$$

($\Rightarrow$) For the forward direction, assume that $\mathscr{P}^* = \mathscr{Q}$. If $\omega(\mathscr{P}) = \infty$ then $\mathscr{P}$ contains all cliques, and by Proposition 6(h) we know that $\mathscr{P} = \mathscr{P}^* = \mathscr{Q}$. Otherwise $\omega(\mathscr{P}) = n - 1 < \infty$ and there exists a set $\mathscr{X}$, such that $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{X})$, where $\mathscr{X} \subseteq \mathrm{Free}(K_n)$ and there are no cliques in $\mathscr{X}$. By Theorem 15, we have

$$\mathscr{P}^* = (\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{X}). \tag{6}$$

But by hypothesis, $\mathscr{P}^* = \mathscr{Q} = \mathrm{Free}(\mathscr{F})$, so we obtain

$$(\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{X}) = \mathrm{Free}(\mathscr{F}). \tag{7}$$

It is clear from (7) that $\mathscr{Q} = \mathrm{Free}(\mathscr{F}) \subseteq (\mathrm{Free}(K_n))^*$, so condition 1 in the inverting perfection theorem is satisfied.

We next show

$$\mathrm{Free}(\mathscr{X}) = \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)). \tag{8}$$

For the forward containment, suppose there is a graph $G \in \mathrm{Free}(\mathscr{X})$ that is *not* in $\mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$. Then there is some $F \in \mathscr{F}$ such that $F \in \mathrm{Free}(K_n)$, and $F \leqslant G$ and $F \in \mathrm{Free}(\mathscr{X})$. But such an F would be in the left-hand side of (7) yet not in the right-hand side, giving a contradiction.

Now suppose that there were a graph $G \in \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$ that is *not* in $\mathrm{Free}(\mathscr{X})$. Thus there exists $X \in \mathscr{X}$ with $X \leqslant G$. We know that $X \in \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$ because G is, yet $K_n \nleqslant X$ (by the assumption that $\mathscr{X} \subseteq \mathrm{Free}(K_n)$), therefore $X \in \mathrm{Free}(\mathscr{F})$. This means that X is in the right-hand side of (7) but not in the left, again giving a contradiction. This establishes (8).

Our progress so far shows that either $\mathscr{P} = \mathscr{Q}$ or else (combining the expression for $\mathscr{P}$ given just above Eq. (6) with Eqs. (8) and (5)) $\mathscr{P}$ has the form

$$\mathscr{P} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{X}) = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)) = \mathrm{Free}(K_n) \cap \mathscr{Q} \quad (9)$$

with $\mathscr{Q} \subseteq (\mathrm{Free}(K_n))^*$. Thus $\mathscr{P}$ has the desired form $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathscr{Q}$ and satisfies condition 1. It remains to show that condition 2 is satisfied.

Note that $\mathscr{F} \cap \mathrm{Free}(K_n)$ does not contain any cliques (since $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ does not) and it is obviously a subset of $\mathrm{Free}(K_n)$, hence Theorem 15 applies and we have

$$\mathscr{P}^* = (\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)). \quad (10)$$

To show that condition 2 is satisfied we must show that for all $F \in \mathscr{F}$, either $F \in \mathrm{Free}(K_n)$ or $F \notin (\mathrm{Free}(K_n))^*$. Suppose (for the sake of a contradiction) that there exists $F \in \mathscr{F}$ which contains a K_n, and is in $(\mathrm{Free}(K_n))^*$. Clearly $F \notin \mathscr{F} \cap \mathrm{Free}(K_n)$. Furthermore, no other graph $F' \in \mathscr{F} \cap \mathrm{Free}(K_n)$ can be induced in F, because $\mathscr{F}$ is the *minimal* set of forbidden graphs for $\mathscr{Q}$ (i.e., if $F_1, F_2 \in \mathscr{F}$, then $F_1 \nleqslant F_2$). Therefore, $F \in \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$, and is in the right-hand side of (10). But $F \notin \mathrm{Free}(\mathscr{F}) = \mathscr{Q} = \mathscr{P}^*$, hence it is not in the left-hand side of (10), giving a contradiction.

($\Leftarrow$) Conversely, we show that if $\mathscr{P}$ has either of the forms given in Theorem 26, then $\mathscr{P}^* = \mathscr{Q}$. If $\mathscr{P} = \mathscr{Q}$ then $\mathscr{P}$ contains all cliques, so by Proposition 6(h), we have $\mathscr{P}^* = \mathscr{P} = \mathscr{Q}$ as desired.

We now consider $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathscr{Q} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$ where $\mathscr{Q} \subseteq (\mathrm{Free}(K_n))^*$ and for all $F \in \mathscr{F}$, either $F \in \mathrm{Free}(K_n)$ or $F \notin (\mathrm{Free}(K_n))^*$. As we have already observed, the set $\mathscr{F} \cap \mathrm{Free}(K_n)$ does not contain any cliques and is a subset of $\mathrm{Free}(K_n)$, so Theorem 15 applies and we have Eq. (10) again. Since $\mathscr{Q} = \mathrm{Free}(\mathscr{F})$, we seek to show

$$(\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)) = \mathrm{Free}(\mathscr{F}). \quad (11)$$

($\supseteq$) By hypothesis, we have $\mathrm{Free}(\mathscr{F}) = \mathscr{Q} \subseteq (\mathrm{Free}(K_n))^*$, and clearly $\mathrm{Free}(\mathscr{F}) \subseteq \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$.

($\subseteq$) Suppose that $G \in (\mathrm{Free}(K_n))^*$ and $G \in \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$ but that $G \notin \mathrm{Free}(\mathscr{F})$. Then there exists $F \in \mathscr{F}$ with $F \leqslant G$. If it were the case that $F \in \mathrm{Free}(K_n)$, then we would have $F \in \mathscr{F} \cap \mathrm{Free}(K_n)$, contradicting the assumption that G is in $\mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$. Otherwise, by our hypothesis (about all graphs in $\mathscr{F}$), it must be the case the $F \notin (\mathrm{Free}(K_n))^*$, contradicting $G \in (\mathrm{Free}(K_n))^*$. Thus equality holds in (11) and the proof is complete. $\square$

Corollary 27. *If $\mathscr{P}$ is a hereditary property with $\mathscr{P}^* = \mathscr{Q}$ and $\omega(\mathscr{P}) = n - 1 < \infty$, then* $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathscr{Q}$.

Proof. If $\mathscr{P}^* = \mathscr{Q}$ then the property $\mathscr{P}$ must have one of the forms given in Theorem 26, namely $\mathscr{P} = \mathscr{Q}$ or $\mathscr{P} = \mathrm{Free}(K_m) \cap \mathscr{Q}$ for some $m \geqslant 2$. Since we are also given

$\omega(\mathscr{P}) = n - 1 < \infty$, we know that $\mathscr{P} \neq \mathscr{Q}$, because $\omega(\mathscr{Q}) = \infty$. Thus $\mathscr{P} = \mathscr{Q}_m = \mathrm{Free}(K_m) \cap \mathscr{Q}$ for some $m \geq 2$. The property $\mathscr{Q}_m$ has $\omega(\mathscr{Q}_m) = m - 1$ for each $m \geq 2$, and therefore we must have $\mathscr{P} = \mathscr{Q}_n = \mathrm{Free}(K_n) \cap \mathscr{Q}$. □

It is tempting to try to use Theorem 26 to resolve the SPGC. However, the following example shows that the inverting perfection theorem is of no help.

Example 28 (*Inverting perfection and the SPGC*). Let $\mathscr{Q}$ be the set of "Berge graphs", that is, $\mathscr{Q}$ is the set of graphs with no induced odd hole or odd anti-hole on 5 or more vertices. The SPGC asserts that $\mathscr{Q} = \{\text{perfect graphs}\}$. In order to test this conjecture, we let $\mathscr{P} = \{\text{edgeless graphs}\}$ and ask whether $\mathscr{P}^* \stackrel{?}{=} \mathscr{Q}$.

Since $\omega(\mathscr{P}) = 1$, we write $\mathscr{P} = \mathrm{Free}(K_2) \cap \mathscr{Q}$ and note that $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ consists of the odd holes and odd anti-holes on 5 or more vertices. By Theorem 26, $\mathscr{P}^* = \mathscr{Q}$ if and only if

1. $\mathscr{Q} \subseteq (\mathrm{Free}(K_2))^*$, and
2. for all $F \in \mathscr{F}$, either $F \in \mathrm{Free}(K_2)$ or $F \notin (\mathrm{Free}(K_2))^*$.

Unfortunately, these two conditions are exactly equivalent to the SPGC itself: the first (which is unresolved) states that all Berge graphs are perfect, and the second (which is easy to see) asserts that all odd holes and odd anti-holes are *not* perfect (clearly $F \notin \mathrm{Free}(K_2)$ for all $F \in \mathscr{F}$).

While the inverting perfection theorem does not tell us whether $\{\text{edgeless graphs}\}^* = \{\text{Berge graphs}\}$, it does tell us exactly which *other* properties $\mathscr{P}$ have $\mathscr{P}^* = \{\text{Berge graphs}\}$.

Example 29 (*Berge graphs*). As in the previous example, let $\mathscr{Q}$ be the set of Berge graphs. If $\mathscr{P} = \mathscr{Q}$ then $\mathscr{P}^* = \mathscr{Q}$ (the trivial case). The other possibility, according to Theorem 26, is $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathscr{Q}$ for some integer $n \geq 2$. We have already discussed the case $n = 2$ in the previous example, so assume $n \geq 3$. Recall that $\mathscr{F} = \{C_{2r+1}: r \geq 2\} \cup \{\overline{C_{2r+1}}: r \geq 2\}$ and consider the odd anti-hole $\overline{C_{2n+1}}$. Clearly $K_n \leqslant \overline{C_{2n+1}}$, so $\overline{C_{2n+1}} \notin \mathrm{Free}(K_n)$. However, $2n + 1 \equiv 3 \not\equiv 1 \bmod (2n - 2)$ because $n \geq 3$, so $\overline{C_{2n+1}} \in (\mathrm{Free}(K_n))^*$ by Theorem 19. This means that condition 2 in Theorem 26 is violated for all $n \geq 3$. Therefore the only properties $\mathscr{P}$ for which $\mathscr{P}^* = \{\text{Berge graphs}\}$ are the trivial case ($\mathscr{P} = \{\text{Berge graphs}\}$) and *possibly* $\mathscr{P} = \{\text{edgeless graphs}\}$ (if and only if the SPGC is true).

4.2. *Applications of the inverting perfection theorem*

In this section we apply the inverting perfection theorem to many specific hereditary properties $\mathscr{Q}$ to find all $\mathscr{P}$ with $\mathscr{P}^* = \mathscr{Q}$. In each case we obtain the trivial solution $\mathscr{P} = \mathscr{Q}$. By Theorem 26, the only other candidates are the properties $\mathscr{Q}_n = \mathrm{Free}(K_n) \cap \mathscr{Q}$. Therefore, the question remaining is to find those $n \geq 2$ for which conditions 1 and 2 of Theorem 26 are satisfied.

Definition 30. For each hereditary property $\mathcal{Q}$ with $\omega(\mathcal{Q}) = \infty$, let $T_{\mathcal{Q}} = \{n \geqslant 2: \mathcal{Q}_n^* = \mathcal{Q}\}$.

In this new notation, Theorem 26 implies that $\mathcal{P}^* = \mathcal{Q}$ if and only if $\mathcal{P} = \mathcal{Q}$ or $\mathcal{P} = \mathcal{Q}_n$ for some $n \in T_{\mathcal{Q}}$. Example 29 tells us that for $\mathcal{Q} = \{\text{Berge graphs}\}$ we have

$$T_{\mathcal{Q}} = \begin{cases} \{2\} & \text{if the SPGC is true,} \\ \emptyset & \text{otherwise.} \end{cases}$$

From now on, our inversion results for the family $\mathcal{Q}$ are given by finding $T_{\mathcal{Q}}$. It is not the purpose of this paper to find $T_{\mathcal{Q}}$ for as many families $\mathcal{Q}$ as possible. Rather, we choose hereditary families $\mathcal{Q}$ with $\omega(\mathcal{Q}) = \infty$ that fall into one or more of the following categories:

1. well-known hereditary families of graphs (e.g., k-colorable of Proposition 43),
2. families that exhibit unusual behavior with respect to the inversion of $*$ (e.g., the family $\mathcal{Q}$ of Proposition 44 for which $T_{\mathcal{Q}} = \emptyset$),
3. families that arise as the set of perfect graphs of a property $\mathcal{P}$ studied in Section 3.

Recall that in Section 3 we discuss analogs of the Strong Perfect Graph Conjecture, that is, we find $\mathcal{P}^*$ for various properties $\mathcal{P}$. Part of our motivation for posing the inverting perfection problem is that once we find $\mathcal{P}^*$ for a hereditary property $\mathcal{P}$ we want to find all other $\hat{\mathcal{P}}$ for which $\hat{\mathcal{P}}^* = \mathcal{P}^*$. Thus we are led to apply the inverting perfection theorem to properties $\mathcal{Q}$ that have arisen as the set of perfect graphs for some family $\mathcal{P}$, i.e., those that fall into the third category.

For families $\mathcal{Q}$ in the third category, we first state the relevant theorem of Section 3 (i.e, we give a property $\mathcal{P}$ for which $\mathcal{P}^* = \mathcal{Q}$) and then we apply the inverting perfection theorem to $\mathcal{Q}$.

For each of the families $\mathcal{Q}$ we have considered, the set $T_{\mathcal{Q}}$ either has $T_{\mathcal{Q}} = \{k, k+1, k+2, \dots\}$ for some $k \geqslant 2$ or has $|T_{\mathcal{Q}}| \leqslant 1$; we study these cases separately.

4.3. Families $\mathcal{Q}$ with $T_{\mathcal{Q}} = \{k, k+1, k+2, \dots\}$

One of the first hereditary properties we consider in Section 3 is $\mathcal{P} = \{\text{acyclic graphs}\}$ for which we find $\mathcal{P}^* = \{\text{chordal graphs}\}$ (see Theorem 10). Now we apply Theorem 26 to the class $\mathcal{Q} = \{\text{chordal graphs}\}$ to find all other properties $\mathcal{P}$ that have $\mathcal{P}^* = \mathcal{Q}$.

Proposition 31 (Chordal). *For the class $\mathcal{Q} = \{\text{chordal graphs}\}$ we have $T_{\mathcal{Q}} = \{3, 4, 5, \dots\}$.*

Proof. Consider $\mathcal{Q}_n = \text{Free}(K_n) \cap \mathcal{Q}$ where $\mathcal{F} = \text{Forb}(\mathcal{Q}) = \{C_r: r \geqslant 4\}$. If $n = 2$ then the cycle $C_4 \in \mathcal{F}$ is *not* in $\text{Free}(K_2)$ but $C_4 \in (\text{Free}(K_2))^* = \{\text{perfect graphs}\}$. This violates one of the conditions of Theorem 26, so $\mathcal{Q}_2^* \neq \mathcal{Q}$. Next let $n \geqslant 3$. In this case, for all $F \in \mathcal{F}$ (i.e., $F \simeq C_r$ for some $r \geqslant 4$) we have $F \in \text{Free}(K_n)$. Furthermore, since chordal graphs are perfect (see Theorem 4.11 in [9]) we have

$$\mathcal{Q} = \{\text{chordal graphs}\} \subseteq \{\text{perfect graphs}\} = (\text{Free}(K_2))^* \subseteq (\text{Free}(K_n))^*,$$

where the last inclusion follows from Theorem 23. Hence the conditions of the inverting perfection theorem hold, and $\mathscr{Q}_n^* = \{\text{chordal graphs}\}$ for $n \geqslant 3$, i.e., $T_{\mathscr{Q}} = \{3,4,5,\dots\}$. Note that in the case $n = 3$, we recover our initial property $\mathscr{Q}_3 = \{\text{acyclic graphs}\}$. □

In Proposition 31 we used the fact that chordal graphs are perfect to satisfy one of the conditions of Theorem 26. The following result generalizes the proof technique used in that instance.

Proposition 32. *Let $\mathscr{Q}$ be a hereditary property with $\omega(\mathscr{Q}) = \infty$ and $\mathscr{Q} \subseteq \{\text{perfect graphs}\}$. Let $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ be the set of forbidden graphs for $\mathscr{Q}$. If there is an integer M so that*

- *there is some graph $H \in \mathscr{F} \cap \{\text{perfect graphs}\}$ with $\omega(H) = M - 1$, and*
- *for all $F \in \mathscr{F}$ we have $\omega(F) \leqslant M - 1$,*

then $T_{\mathscr{Q}} = \{M, M+1, M+2, \dots\}$.

Note that if we let $\mathscr{Q} = \{\text{chordal graphs}\}$, $H = C_4$, and $M = 3$, as in Proposition 31, then the hypotheses of Proposition 32 are satisfied. Indeed the conclusions of Propositions 31 and 32 agree.

Proof of Proposition 32. Let the property $\mathscr{Q}$, the integer M and the graph $H \in \mathrm{Forb}(\mathscr{Q})$ satisfy the hypotheses of Proposition 32. We consider the property $\mathscr{Q}_n = \mathrm{Free}(K_n) \cap \mathscr{Q}$ for $n \geqslant 2$ and show that it satisfies the conditions of the inverting perfection theorem in the case $n \geqslant M$, but *not* in the case $n \leqslant M - 1$.

If $n \leqslant M - 1$ then the graph H (with $\omega(H) = M - 1 \geqslant n$) is *not* in $\mathrm{Free}(K_n)$. However, $H \in \{\text{perfect graphs}\} = (\mathrm{Free}(K_2))^* \subseteq (\mathrm{Free}(K_n))^*$ by Theorem 23. This violates the second condition of the inverting perfection theorem, hence $\mathscr{Q}_n^* \neq \mathscr{Q}$ and $n \notin T_{\mathscr{Q}}$ for $n \leqslant M - 1$.

If $n \geqslant M$ then for all $F \in \mathscr{F}$, we have $\omega(F) \leqslant M - 1 < n$, and thus $F \in \mathrm{Free}(K_n)$. In the hypothesis we assumed that $\mathscr{Q} \subseteq \{\text{perfect graphs}\} = (\mathrm{Free}(K_2))^*$ so $\mathscr{Q} \subseteq (\mathrm{Free}(K_n))^*$ for all $n \geqslant 2$ by Theorem 23. Hence for $n \geqslant M$ the conditions of the inverting perfection theorem are satisfied; so $\mathscr{Q}_n^* = \mathscr{Q}$ and $n \in T_{\mathscr{Q}}$ for $n \geqslant m$. □

Proposition 33 (Interval graphs). *For the class $\mathscr{Q} = \{\text{interval graphs}\}$ we have $T_{\mathscr{Q}} = \{5,6,7,\dots\}$.*

Proof. The class of interval graphs is a well-known family of perfect graphs (see [9] for definitions and a proof that interval graphs are perfect). It is easy to see that $\omega(\mathscr{Q}) = \infty$. The graph $H \in \mathrm{Forb}(\mathscr{Q})$ (shown in Fig. 2) is perfect and has $\omega(H) = 4$. Furthermore, *all* graphs $F \in \mathrm{Forb}(\mathscr{Q})$ have $\omega(F) \leqslant 4$[7]. Therefore Proposition 32 applies and we conclude that $T_{\mathscr{Q}} = \{5,6,7,\dots\}$. □

Proposition 34 (Cochromatic number). *For the hereditary class $\mathscr{Q} = \{K_m\colon m \geqslant 1\} \cup \{\overline{K_m}\colon m \geqslant 1\}$ we have $T_{\mathscr{Q}} = \{3,4,5,\dots\}$.*

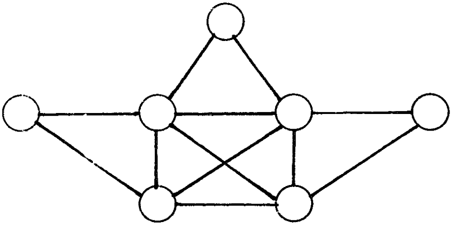

Fig. 2. A graph $H \in \mathrm{Forb}(\{\text{interval graphs}\})$ with $\omega(H) = 4$.

Proof. In this example we have $\mathscr{Q} \subseteq \{\text{perfect graphs}\}$ and the set of forbidden graphs for $\mathscr{Q}$ is finite, namely $\mathrm{Forb}(\mathscr{Q}) = \{P_3, \overline{P_3}\}$. Since both P_3 and $\overline{P_3}$ are perfect graphs, and $\omega(P_3) = \omega(\overline{P_3}) = 2$, Proposition 32 implies that $T_{\mathscr{Q}} = \{3, 4, 5, \dots\}$. □

The next proposition gives a family $\mathscr{Q}$ for which *every* possible candidate is indeed included in $\mathscr{T}_{\mathscr{Q}}$. On the other extreme, Proposition 44 shows that the set $T_{\mathscr{Q}}$ can be empty.

Proposition 35. *For the family* $\mathscr{Q} = \{\textit{all cliques}\}$ *we have* $T_{\mathscr{Q}} = \{2, 3, 4, \dots\}$.

Proof. Since $\mathscr{Q} \subseteq \{\text{perfect graphs}\} \subseteq (\mathrm{Free}(K_n))^*$ for all $n \geqslant 2$, by Theorem 23, condition 1 of the inverting perfection theorem holds. The set of minimal forbidden graphs for $\mathscr{Q}$ has a particularly simple form, namely $\mathrm{Forb}(\mathscr{Q}) = \{\overline{K_2}\}$. Since $\overline{K_2} \in \mathrm{Free}(K_n)$ for all $n \geqslant 2$ we know that condition 2 is also satisfied. Therefore $T_{\mathscr{Q}} = \{2, 3, 4, \dots\}$. □

Since we are applying the inverting perfection theorem to families $\mathscr{Q} = \mathscr{P}^*$ for the properties $\mathscr{P}$ discussed in Section 3, we should take advantage of the fact that each of those families $\mathscr{P}$ is unit-based (see Definition 9).

Lemma 36. *If* $\mathscr{Q} = \mathscr{P}^*$ *where* $\mathscr{P}$ *is a unit-based property with* $\omega(\mathscr{P}) < \infty$, *then condition* 2 *of Theorem* 26 *is satisfied for all* $n > \omega(\mathscr{P})$. *In particular, for all graphs* $F \in \mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ *we have* $F \in \mathrm{Free}(K_n)$ *for* $n > \omega(\mathscr{P})$.

Proof. Let F be a graph in $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ which means that F is a minimal $\chi_{\mathscr{P}}$-imperfect graph. Since $\mathscr{P}$ is a unit-based property, we know that $\omega_{\mathscr{P}}(F) = 1$. Therefore

$$1 = \omega_{\mathscr{P}}(F) = \left\lceil \frac{\omega(F)}{\omega(\mathscr{P})} \right\rceil \geqslant \left\lceil \frac{\omega(F)}{n-1} \right\rceil$$

for all integers $n > \omega(\mathscr{P})$. So we have $\omega(F) \leqslant n - 1$ and consequently $F \in \mathrm{Free}(K_n)$. □

Theorem 12 states that the property $\mathscr{P} = \{G\colon \Delta(G) \leqslant t\}$ is unit-based for all $t \geqslant 1$ and that $\mathscr{P}^* = \{G\colon \text{for every } v \in V(G) \text{ either } d(v) \leqslant t \text{ or } v \text{ is simplicial}\}$. The following proposition applies the inverting perfection theorem to $\mathscr{P}^*$.

Proposition 37 (Bounded maximum degree). *If $\mathscr{P} = \{G\colon \Delta(G) \leqslant t\}$ with $t \geqslant 1$ a fixed integer, and $\mathscr{Q} = \mathscr{P}^*$, then $T_{\mathscr{Q}} = \{t+2, t+3, t+4, \dots\}$.*

We omit the proofs of Propositions 37 and 38 due to space constraints; they can be found in [20].

By Theorem 11, the property $\mathscr{P} = \{\text{unicyclic graphs}\} = \{G\colon G \text{ has at most 1 cycle}\}$ is unit-based and the set Forb($\mathscr{P}$) is shown in Fig. 1.

Proposition 38 (Unicyclic). *If $\mathscr{P} = \{\text{unicyclic graphs}\}$ and $\mathscr{Q} = \mathscr{P}^*$ then $T_{\mathscr{Q}} = \{4, 5, 6, \dots\}$.*

Each of the properties $\mathscr{Q}$ considered in this section has $T_{\mathscr{Q}} = \{k, k+1, \dots\}$ for some $k \geqslant 2$. While we have not been able to specify exactly which $T_{\mathscr{Q}}$ have the form $\{k, k+1, \dots\}$, the following is a partial result.

Proposition 39. *If $\mathscr{P}$ is a unit-based property with $\omega(\mathscr{P}) < \infty$ and $\mathscr{P}^* = \mathscr{Q}$, then $|T_{\mathscr{Q}}| = \infty$.*

Proof. Let $\omega(\mathscr{P}) = n - 1 < \infty$ and let $\mathscr{F} = \text{Forb}(\mathscr{Q})$. Then $\mathscr{P} = \mathscr{Q}_n = \text{Free}(K_n) \cap \mathscr{Q}$ where $\mathscr{Q} \subseteq (\text{Free}(K_n))^*$ and for all $F \in \mathscr{F}$, either $F \in \text{Free}(K_n)$ or $F \notin (\text{Free}(K_n))^*$ (Theorem 26 and Corollary 27). Consider $\mathscr{Q}_m = \text{Free}(K_m) \cap \mathscr{Q}$ for any integer m with $(n-1) | (m-1)$. Then $(\text{Free}(K_n))^* \subseteq (\text{Free}(K_m))^*$ by Theorem 23, so $\mathscr{Q} \subseteq (\text{Free}(K_m))^*$. This is the first condition of Theorem 26. By Lemma 36, condition 2 of Theorem 26 is satisfied for all $m > n - 1$. Hence both conditions of Theorem 26 are satisfied, and $\mathscr{Q}_m^* = \mathscr{Q}$ for all m with $(n-1)|(m-1)$. Thus $k(n-1) + 1 \in T_{\mathscr{Q}}$ for all $k \geqslant 1$, and in particular, $|T_{\mathscr{Q}}| = \infty$. $\square$

4.4. Families $\mathscr{Q}$ with $|T_{\mathscr{Q}}| \leqslant 1$

The families $\mathscr{Q}$ we considered in the previous section all had $T_{\mathscr{Q}} = \{k, k+1, k+2, \dots\}$ for some k. We now apply the inverting perfection theorem to classes $\mathscr{Q}$ for which $|T_{\mathscr{Q}}| \leqslant 1$. The important classes $\mathscr{Q} = (\text{Free}(K_n))^*$ for $n \geqslant 2$ fall into this category.

Lemma 40. *If $2 < m < n$ then there exists an integer $r \geqslant 2n + 1$ such that $C_r \in (\text{Free}(K_n))^*$ but $\overline{C_r} \notin (\text{Free}(K_m))^*$.*

Note that the following proof of Lemma 40 also provides a proof of the forward direction of Theorem 23.

Proof of Lemma 40. Since $m < n$ we have $(n-1)\nmid(m-1)$. Let $r = (2m-2)q+1$ where $q \geqslant \max\{n,m\}$ is a prime number. Clearly $r \geqslant (2m-2)n+1 \geqslant 2n+1$ and $r \geqslant (2m-2)n+1 > 2m-1$ since $n,m > 2$. But $r \equiv 1 \bmod (2m-2)$ and $r > 2m-1$, so we know that $\overline{C_r} \notin (\mathrm{Free}(K_m))^*$ by Theorem 19.

Now let $r' \equiv r \bmod (2n-2)$, with r' an odd integer between 1 and $2n-3$, inclusive. If $r' = 1$, then $(2n-2)|(2m-2)q$, or equivalently, $(n-1)|(m-1)q$. Note that $n-1$ and q are relatively prime, since $n-1 < q$ and q is prime. Hence $(n-1)|(m-1)$, a contradiction. But then it must be the case that $3 \leqslant r' \leqslant 2n-3$. Therefore, if we let $s = n-(r'+1)/2$, we have $1 \leqslant s \leqslant n-1$. By Theorem 21, the graph $\overline{C_r} \vee K_s$ is a minimal χ_n-imperfect graph, and therefore $\overline{C_r}$ is χ_n-perfect. So $\overline{C_r}$ is in $(\mathrm{Free}(K_n))^*$ but not in $(\mathrm{Free}(K_m))^*$. $\square$

Proposition 41. *If $\mathscr{Q}$ is a class for which $\overline{C_s} \in \mathscr{Q}$ for infinitely many $s \geqslant 4$, then $|T_{\mathscr{Q}}| \leqslant 1$.*

Proof. Let $\mathscr{Q} = \mathrm{Free}(\mathscr{F})$ be a class containing infinitely many anti-holes $\overline{C_s}$. For the sake of a contradiction, assume that $|T_{\mathscr{Q}}| \geqslant 2$ and let $m, n \in T_{\mathscr{Q}}$ with $m < n$. By the definition of $T_{\mathscr{Q}}$ we have $\mathscr{Q}_m^* = \mathscr{Q}_n^* = \mathscr{Q}$ where

$$\mathscr{Q}_m = \mathrm{Free}(K_m) \cap \mathscr{Q} = \mathrm{Free}(K_m) \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_m)),$$

$$\mathscr{Q}_n = \mathrm{Free}(K_n) \cap \mathscr{Q} = \mathrm{Free}(K_n) \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)).$$

By Theorem 15 we know that

$$\mathscr{Q}_m^* = (\mathrm{Free}(K_m))^* \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_m)),$$

$$\mathscr{Q}_n^* = (\mathrm{Free}(K_n))^* \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n)).$$

There exists $r \geqslant 2n+1$ such that $\overline{C_r} \in (\mathrm{Free}(K_n))^*$ but $\overline{C_r} \notin (\mathrm{Free}(K_m))^*$ by Lemma 40. Therefore $\overline{C_r} \notin \mathscr{Q}_m^* = \mathscr{Q}_n^* = \mathscr{Q}$, and consequently, $\overline{C_r} \notin \mathscr{Q}_n^* = (\mathrm{Free}\ (K_n))^* \cap \mathrm{Free}(\mathscr{F} \cap \mathrm{Free}(K_n))$. Since $\overline{C_r} \in (\mathrm{Free}(K_n))^*$, there must be some graph $F \in \mathscr{F} \cap \mathrm{Free}(K_n)$ with $F \leqslant \overline{C_r}$. We cannot have $F = \overline{C_r}$, because $r \geqslant 2n+1$ and thus $K_n \leqslant \overline{C_r}$. This means that $F < \overline{C_r}$ or equivalently $F \leqslant \overline{P_r}$. In this case, $F \leqslant \overline{P_R} < \overline{C_R}$ for all $R \geqslant r$ and thus $\overline{C_R} \notin \mathrm{Free}(\mathscr{F}) = \mathscr{Q}$ for all $R \geqslant r$. This contradicts the assumption that $\mathscr{Q}$ contains infinitely many anti-holes; thus $|T_{\mathscr{Q}}| \leqslant 1$ as desired. $\square$

Corollary 42 ($\mathrm{Free}(K_n)$). *If $\mathscr{Q} = (\mathrm{Free}(K_n))^*$ where $n \geqslant 2$ then $T_{\mathscr{Q}} = \{n\}$.*

Proof. Let $\mathscr{Q} = (\mathrm{Free}(K_n))^*$ for some integer $n \geqslant 2$. Then Theorem 19 tells us that there are infinitely many anti-holes $\overline{C_s}$ contained in $\mathscr{Q}$ (for instance, all the even anti-holes are in $\mathscr{Q}$). Therefore we apply Proposition 41 and conclude that $|T_{\mathscr{Q}}| \leqslant 1$. Since the property $\mathscr{Q}_n = \mathrm{Free}(K_n) = \mathrm{Free}(K_n) \cap \mathscr{Q}$ has $\mathscr{Q}_n^* = \mathscr{Q}$ and $\omega(\mathscr{Q}_n) = n-1$, Corollary 27 implies that $n \in T_{\mathscr{Q}}$. Combining the results $|T_{\mathscr{Q}}| \leqslant 1$ and $n \in T_{\mathscr{Q}}$ we obtain $T_{\mathscr{Q}} = \{n\}$. $\square$

The best known instance of Corollary 42 is the case $n = 2$ for which $\mathscr{Q} = \{\text{perfect graphs}\}$. The corollary tells us that $T_{\mathscr{Q}} = \{2\}$, that is, the only property $\mathscr{P}$ with $\omega(\mathscr{P}) < \infty$ and $\mathscr{P}^* = \{\text{perfect graphs}\}$ is $\mathscr{P} = \{\text{edgeless graphs}\}$. In Example 29 we saw that the property $\mathscr{Q}' = \{\text{Berge graphs}\}$ has $T_{\mathscr{Q}'} = \{2\}$ if and only if the SPGC holds. The SPGC asserts that $\mathscr{Q} = \mathscr{Q}'$ and thus if the SPGC is true we should have $T_{\mathscr{Q}} = T_{\mathscr{Q}'}$, which indeed we do.

Proposition 43 (k-colorable). *If $\mathscr{P} = \{k\text{-colorable}\} = \{G: \chi(G) \leqslant k\}$ for some $k \geqslant 1$, and $\mathscr{Q} = \mathscr{P}^*$ then $T_{\mathscr{Q}} = \{k + 1\}$.*

Proof. For the property $\mathscr{P} = \{k\text{-colorable}\}$ we have $\omega(\mathscr{P}) = k$. Thus $\omega_{\mathscr{P}}(G) = \lceil \omega(G)/k \rceil$ and $\chi_{\mathscr{P}}(G) = \lceil \chi(G)/k \rceil$. This means that all perfect graphs are $\chi_{\mathscr{P}}$-perfect, and in particular $\overline{C_{2r}} \in \mathscr{P}^* = \mathscr{Q}$ for each $r \geqslant 2$ (see Lemma 19). So we have an infinite family of anti-holes in $\mathscr{Q}$ and therefore Proposition 41 implies that $|T_{\mathscr{Q}}| \leqslant 1$. Since $\mathscr{P}^* = \mathscr{Q}$ with $\omega(\mathscr{P}) = k$, we know that $k + 1 \in T_{\mathscr{Q}}$ by Corollary 27, and thus $T_{\mathscr{Q}} = \{k + 1\}$. $\square$

For $k = 1$, the property $\mathscr{P} = \{1\text{-colorable}\} = \{\text{edgeless graphs}\}$ and $\mathscr{Q} = \{\text{perfect graphs}\}$. Thus Proposition 43 gives another proof that $T_{\mathscr{Q}} = \{2\}$ where $\mathscr{Q} = \{\text{perfect graphs}\}$. In the case $k = 2$, the property $\mathscr{P}$ is $\{2\text{-colorable}\} = \{\text{bipartite graphs}\}$ and Proposition 43 implies that for $\mathscr{Q} = \mathscr{P}^*$ we have $T_{\mathscr{Q}} = \{3\}$. Thus the only property $\hat{\mathscr{P}}$ with $\hat{\mathscr{P}}^* = \{\text{bipartite graphs}\}^*$ and $\omega(\hat{\mathscr{P}}) < \infty$ is $\hat{\mathscr{P}} = \{\text{bipartite graphs}\}$.

The next proposition gives an example of a property $\mathscr{Q}$ whose only preimage under $*$ is itself.

Proposition 44. *If $\mathscr{Q}$ is the class given by $\mathscr{Q} = \text{Free}(\mathscr{F})$ with $\mathscr{F} = \{\overline{C_r}: r \geqslant 5\}$ then $T_{\mathscr{Q}} = \emptyset$, that is, $\mathscr{P}^* = \mathscr{Q} \Rightarrow \mathscr{P} = \mathscr{Q}$.*

Proof. First consider $\mathscr{Q}_2 = \text{Free}(K_2) \cap \mathscr{Q}$. The cycle C_7 is in $\mathscr{Q}$, but $C_7 \notin \{\text{perfect graphs}\} = (\text{Free}(K_2))^*$. Therefore $\mathscr{Q} \nsubseteq (\text{Free}(K_2))^*$; so Theorem 26 implies that $2 \notin T_{\mathscr{Q}}$. For $n \geqslant 3$, the anti-hole $\overline{C_{2n+1}} \in \mathscr{F}$ violates condition 2 of the inverting perfection theorem, because $\overline{C_{2n+1}} \in (\text{Free}(K_n))^*$ by Lemma 19 and $\overline{C_{2n+1}} \notin \text{Free}(K_n)$. Therefore $n \notin T_{\mathscr{Q}}$ for $n \geqslant 3$. We conclude that $T_{\mathscr{Q}} = \emptyset$. $\square$

We have seen three different categories of properties $\mathscr{Q}$ in the range of $*$; those with $T_{\mathscr{Q}} = \{k, k + 1, \dots\}$ for some $k \geqslant 2$, those with $T_{\mathscr{Q}} = \{k\}$ for some $k \geqslant 2$, and those with $T_{\mathscr{Q}} = \emptyset$. We have not been able to find *any* properties $\mathscr{Q}$ in the range of $*$ that do not fall into one of these categories, which leads us to the following question:

Question 45. Which subsets of $\{2, 3, 4, \dots\}$ can be realized as $T_{\mathscr{Q}}$ for some hereditary family $\mathscr{Q}$ containing all cliques?

4.5. Another $\mathscr{X}$-free restriction theorem

In Theorem 15 we found $\hat{\mathscr{P}}^*$ in terms of $\mathscr{P}^*$, where $\hat{\mathscr{P}}$ is an $\mathscr{X}$-free restriction of $\mathscr{P}$. In this inverse version we get information about $T_{\mathcal{Q}'}$ from knowing $T_{\mathcal{Q}}$, where $\mathcal{Q}'$ is an $\mathscr{X}$-free restriction of $\mathcal{Q}$.

Theorem 46. *Let $\mathcal{Q}$ be a hereditary class with $\omega(\mathcal{Q}) = \infty$, and let $\mathscr{X}$ be a set with $\mathscr{X} \subseteq \mathrm{Free}(K_n) \cap \mathcal{Q}$ for all $n \in T_{\mathcal{Q}}$ and $K_m \notin \mathscr{X}$ for all m. If $\mathcal{Q}' = \mathcal{Q} \cap \mathrm{Free}(\mathscr{X})$ then $T_{\mathcal{Q}} \subseteq T_{\mathcal{Q}'}$.*

Proof. Suppose $n \in T_{\mathcal{Q}}$. Thus the property $\mathcal{Q}_n = \mathrm{Free}(K_n) \cap \mathcal{Q}$ has $\mathcal{Q}_n^* = \mathcal{Q}$. If we write $\hat{\mathscr{P}}_n = \mathcal{Q}_n \cap \mathrm{Free}(\mathscr{X})$ we note that $\mathscr{X} \subseteq \mathrm{Free}(K_n) \cap \mathcal{Q} = \mathcal{Q}_n$ and $K_m \notin \mathscr{X}$ for all m. Thus the conditions of Theorem 15 are satisfied and

$$\hat{\mathscr{P}}^*_n = \mathcal{Q}_n^* \cap \mathrm{Free}(\mathscr{X}) = \mathcal{Q} \cap \mathrm{Free}(\mathscr{X}) = \mathcal{Q}'.$$

Since $\omega(\hat{\mathscr{P}}_n) = \omega(\mathcal{Q}_n) = n - 1$, we have $n \in T_{\mathcal{Q}'}$. $\square$

Example 47 (*Threshold graphs*). Let $\mathcal{Q} = \{$chordal graphs$\}$ and let $\mathcal{Q}' = \{$threshold graphs$\}$, that is $\mathcal{Q}' = \mathcal{Q} \cap \mathrm{Free}(\mathscr{X})$ where $\mathscr{X} = \{P_4, 2K_2\}$ [9]. Since $\mathscr{X} \subseteq \mathrm{Free}(K_n) \cap \mathcal{Q}$ for all $n \geqslant 3$ and there are no cliques in $\mathscr{X}$ we have $T_{\mathcal{Q}} \subseteq T_{\mathcal{Q}'}$ by Theorem 46. Thus $\{3, 4, 5, \ldots\} \subseteq T_{\mathcal{Q}'}$ and we need only check whether $2 \overset{?}{\in} T_{\mathcal{Q}'}$. The path $P_4 \in \mathrm{Forb}(\mathcal{Q}')$ is perfect (i.e., $P_4 \in (\mathrm{Free}(K_2))^*$) yet is not in $\mathrm{Free}(K_2)$, hence $2 \notin T_{\mathcal{Q}'}$ by Theorem 26 and hence $T_{\mathcal{Q}'} = \{3, 4, 5, \ldots\}$.

Example 47 is meant to illustrate the use of Theorem 46. In fact it is easier to find $T_{\mathcal{Q}'}$ for, $\mathcal{Q}' = \{$threshold graphs$\}$ by a direct application of Proposition 32. The following example shows that the inequality $T_{\mathcal{Q}} \subseteq T_{\mathcal{Q}'}$ in Theorem 46 can be strict.

Example 48. Let $\mathcal{Q} = \{K_m\colon m \geqslant 1\} \cup \{\overline{K_m}\colon m \geqslant 1\}$ and $\mathcal{Q}' = \{$all cliques$\}$. Then we can write $\mathcal{Q}' = \mathcal{Q} \cap \mathrm{Free}(\mathscr{X})$ where $\mathscr{X} = \{\overline{K_2}\}$. Clearly $K_m \notin \mathscr{X}$ for all m, and $\mathscr{X} \subseteq \mathrm{Free}(K_n) \cap \mathcal{Q}$ for all $n \geqslant 2$ (and therefore for all $n \in T_{\mathcal{Q}}$). Hence Theorem 46 applies and tells us that $T_{\mathcal{Q}} \subseteq T_{\mathcal{Q}'}$. In fact the inequality is strict because in Propositions 34 and 35 we found that $T_{\mathcal{Q}} = \{3, 4, 5, \ldots\}$ and $T_{\mathcal{Q}'} = \{2, 3, 4, \ldots\}$.

4.6 Double inversion

For a fixed family $\mathcal{Q}$, the inverting perfection problem is to find which properties $\mathscr{P}$ have $\mathscr{P}^* = \mathcal{Q}$. In the previous section we saw that the solution to this problem can be given via the set $T_{\mathcal{Q}}$ where $\mathscr{P}^* = \mathcal{Q}$ if and only if $\mathscr{P} = \mathcal{Q}$ or $\mathscr{P} = \mathrm{Free}(K_n) \cap \mathcal{Q}$ for some $n \in T_{\mathcal{Q}}$. Thus there is an "inverting perfection" function $\mathrm{IP}\colon \mathcal{Q} \mapsto T_{\mathcal{Q}}$, whose domain is the set of hereditary properties containing all cliques and whose range is the set of subsets of $\{2, 3, 4, \ldots\}$.

In this new notation, Question 45 can be rephrased as: "what is the image of the function IP?". A more general question is that of inverting the function IP.

Double inversion problem. Given a set $T \subseteq \{2,3,4,\ldots\}$, find all hereditary families $\mathscr{Q}$ with $\omega(\mathscr{Q}) = \infty$ and $T_{\mathscr{Q}} = T$.

From the examples in the preceding sections, we see that $T_{\mathscr{Q}} = \{3,4,5,\ldots\}$ for the properties $\mathscr{Q} = \{\text{chordal graphs}\}$, $\mathscr{Q} = \{\text{threshold graphs}\}$, $\mathscr{Q} = \{G\colon \Delta(G) \leqslant 1\}$, and $\mathscr{Q} = \{K_m\colon m \geqslant 1\} \cup \{\overline{K_m}\colon m \geqslant 1\}$. It is not obvious that these four classes should have the same $T_{\mathscr{Q}}$ set, nor are they the only classes with $T_{\mathscr{Q}} = \{3,4,5,\ldots\}$.

The following theorem solves the double inversion problem in the case that $2 \in T$.

Theorem 49. *Let $T \subseteq \{2,3,4,\ldots\}$ with $2 \in T$.*

- *If $T = \{2\}$, then $T_{\mathscr{Q}} = T \Leftrightarrow \mathscr{Q} = \{\text{perfect graphs}\}$ or $\mathscr{Q} = \{\text{perfect graphs}\} \cap \mathrm{Free}(\overline{K_n})$ for some $n \geqslant 2$.*
- *If $T = \{2,3,4,\ldots\}$, then $T_{\mathscr{Q}} = T \Leftrightarrow \mathscr{Q} = \{\text{cliques}\}$.*
- *For all other $T \subseteq \{2,3,4,\ldots\}$ with $2 \in T$, there is no hereditary family $\mathscr{Q}$ containing all cliques with $T_{\mathscr{Q}} = T$.*

Proof. Let $T \subseteq \{2,3,4,\ldots\}$ with $2 \in T$, and suppose that there exists a hereditary property $\mathscr{Q}$ with $\omega(\mathscr{Q}) = \infty$ and $T_{\mathscr{Q}} = T$. Thus $2 \in T_{\mathscr{Q}}$ and by Theorem 26 we have

1. $\mathscr{Q} \subseteq (\mathrm{Free}(K_2))^* = \{\text{perfect graphs}\}$.
2. For all $F \in \mathscr{F} = \mathrm{Forb}(\mathscr{Q})$, either $F \in \mathrm{Free}(K_2) = \{\overline{K_1}, \overline{K_2}, \overline{K_3}, \ldots\}$, or $F \notin (\mathrm{Free}(K_2))^* = \{\text{perfect graphs}\}$.

Case 1: *$\overline{K_n} \notin \mathscr{F}$ for all $n \geqslant 2$.* Above we saw that $\mathscr{Q} \subseteq \{\text{perfect graphs}\}$. In this case, we show that in fact $\mathscr{Q} = \{\text{perfect graphs}\}$. If not, there is some perfect graph G which is not in $\mathscr{Q}$. By the definition of $\mathscr{F}$, there exists $F \in \mathscr{F}$ with $F \leqslant G$. But this is impossible since G is perfect and F is not. Thus $\mathscr{Q} = \{\text{perfect graphs}\}$, and $T_{\mathscr{Q}} = \{2\}$ by Corollary 42.

Case 2: *$\overline{K_n} \in \mathscr{F}$ for some $n \geqslant 2$.* Note that there can be no other edgeless graph (besides $\overline{K_n}$) in $\mathscr{F}$, because if $\overline{K_m} \in \mathscr{F}$ with $m \neq n$, then either $\overline{K_n} \leqslant \overline{K_m}$ or $\overline{K_m} \leqslant \overline{K_n}$. This would contradict the definition of $\mathscr{F} = \mathrm{Forb}(\mathscr{Q})$ as the *minimal* set of forbidden graphs for $\mathscr{Q}$.

If $n = 2$ (i.e., $\overline{K_2} \in \mathscr{F}$), then $\mathscr{Q} \subseteq \{\text{cliques}\}$. However, $\{\text{cliques}\} \subseteq \mathscr{Q}$ by the assumption $\omega(\mathscr{Q}) = \infty$. Hence $\mathscr{Q} = \{\text{cliques}\}$, and $T_{\mathscr{Q}} = \{2,3,4,\ldots\}$ by Proposition 35.

Now assume $n \geqslant 3$. Recall that for all $F \in \mathscr{F}$ either F is an edgeless graph or F is *not* perfect. By the assumptions of this case, the only edgeless graph in $\mathscr{F}$ is $\overline{K_n}$ for some $n \geqslant 3$, so all other graphs in $\mathscr{F}$ are not perfect.

We know that $\mathscr{Q} \subseteq \{\text{perfect graphs}\}$ and further that $\mathscr{Q} \subseteq \mathrm{Free}(\overline{K_n})$ since $\overline{K_n} \in \mathscr{F}$, hence

$$\mathscr{Q} \subseteq \{\text{perfect graphs}\} \cap \mathrm{Free}(\overline{K_n}). \tag{12}$$

To show that reverse inclusion holds in (12), let G be a graph in the right-hand side, that is, G is perfect and $\overline{K_n} \not\leqslant G$. For a contradiction, suppose that $G \notin \mathscr{Q}$, so that there exists a graph $F \in \mathscr{F}$ with $F \leqslant G$. Therefore F is perfect and $\overline{K_n} \nleq F$, which contradicts the fact that the only perfect graph in $\mathscr{F}$ is $\overline{K_n}$.

Thus $\mathscr{Q} = \{\text{perfect graphs}\} \cap \text{Free}(\overline{K_n})$ for some $n \geqslant 3$. Note that $\overline{C_{2r}} \in \mathscr{Q}$ for all $r \geqslant 2$, hence $|T_{\mathscr{Q}}| \leqslant 1$ by Proposition 41. It is easy to check that indeed $2 \in T_{\mathscr{Q}}$ (using Theorem 26), hence $T_{\mathscr{Q}} = \{2\}$. □

5. Open questions

We discuss some open problems related to characterizing the $\chi_{\mathscr{P}}$-perfect graphs for hereditary properties $\mathscr{P}$ and to inverting perfection. A more complete listing of open problems in generalized $\chi_{\mathscr{P}}$-perfect graphs is given in [20].

(1) In Section 3 we find $\mathscr{P}^*$ for a number of hereditary properties $\mathscr{P}$. For which additional hereditary properties $\mathscr{P}$ can we find $\mathscr{P}^*$? In particular, can we find $\mathscr{P}^*$ for a property that is not unit-based?

(2) For a hereditary property $\mathscr{P}$, let $S_{\mathscr{P}} = \{k$: there exists a minimal $\chi_{\mathscr{P}}$-imperfect graph with $\omega_{\mathscr{P}}(G) = k\}$. If $\mathscr{P}$ is unit-based then by definition, $S_{\mathscr{P}} = \{1\}$. For the property $\mathscr{P} = \{\text{edgeless graphs}\}$, we know that the odd anti-hole $\overline{C_{2r+1}}$ is a minimal ($\chi_{\mathscr{P}}$-) imperfect graphs with $\omega(\overline{C_{2r+1}}) = r$ for each $r \geqslant 2$, and there are no minimal imperfect graphs with $\omega = 1$, therefore $S_{\mathscr{P}} = \{2, 3, 4, \ldots\}$. It is not hard to check that for $\mathscr{P} = \{\text{bipartite graphs}\}$ we have $S_{\mathscr{P}} = \{1, 2, 3, \ldots\}$ (write $\mathscr{P} = \text{Free}(K_3) \cap \text{Free}(\mathscr{X})$, apply Theorem 15, and note that the cycle C_5 and the anti-holes $\overline{C_{4r+1}}$ for $r \geqslant 2$ are minimal $\chi_{\mathscr{P}}$-imperfect graphs). However these three are the *only* sets we have found to arise as $S_{\mathscr{P}}$ sets. It is open to either prove that these are the only possibilities or to find an example of a hereditary property $\mathscr{P}$ with a different $S_{\mathscr{P}}$ set.

(3) In Section 3 we have some partial results in characterizing $(\text{Free}(K_n))^*$, including Theorem 21 which finds a family of minimal χ_n-imperfect graphs with arbitrarily large clique size. What additional minimal χ_n-imperfect graphs can we find? In particular, can we find any minimal χ_n-imperfect graphs (for $n \geqslant 3$) which do *not* contain any odd anti-holes as induced subgraphs?

(4) For many of the properties $\mathscr{Q}$ we encountered in Section 4, it is not too difficult to find $T_{\mathscr{Q}}$. In fact, we found $T_{\mathscr{Q}}$ for $\mathscr{Q} = (\text{Free}(K_n))^*$ (Corollary 42) while we are unable even to find the set $\mathscr{Q}$ itself. For other families $\mathscr{Q}$, notably the set $\mathscr{Q} = \{\text{Berge graphs}\}$, it appears to be quite difficult. Can we quantify what it is about $\mathscr{Q}$ that makes finding $T_{\mathscr{Q}}$ more difficult in some cases than in others?

(5) In Section 4 we formulated the double inversion problem: given a set $T \subseteq \{2, 3, 4, \ldots\}$, find all hereditary families $\mathscr{Q}$ with $\omega(\mathscr{Q}) = \infty$ and $T_{\mathscr{Q}} = T$. We are also interested in solving some weaker forms of this question:

- For which sets T is there a solution to the double inversion problem (Question 45)?
- Which properties $\mathscr{Q}$ have $|T_{\mathscr{Q}}| = \infty$?
- Which properties $\mathscr{Q}$ have $3 \in T_{\mathscr{Q}}$?

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